Composable Efficient Array Computations Using Linear Types

APPENDIX

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Abstract

Functional languages excel at describing complex programs as the composition of small building blocks. Yet, the performance properties of such compositions depend on compiler heuristics, whose behavior is difficult to predict.

In this paper, we introduce an array-programming calculus which can guarantee elimination of intermediate arrays when composing two programs, eliminating the cost of composition. The calculus has linear types and is an extension of Girard's linear logic with vector types and a synchronization primitive.

We illustrate the effectiveness of our language by implementing a number of classical algorithms in a compositional style, and then compiling these examples to efficient code.

Keywords Array-Programming, Classical Linear Logic

1. Cut-elimination: proof cases

Principal reduction rules.

$$\dfrac{\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \Delta, B^\perp \vdash \mathrm{b}}{\Gamma, \Delta, A^\perp \parr B^\perp \vdash}\parr \qquad \dfrac{\Xi, A, B \vdash \mathrm{c}}{\Xi, A \otimes B \vdash}\otimes}{\Gamma, \Delta, \Xi \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, B^\perp \vdash \mathrm{b} \qquad \Xi, A, B \vdash \mathrm{c}}{\Delta, \Xi, A \vdash}\textsc{Fuse}}{\Gamma, \Delta, \Xi \vdash}\textsc{Fuse}$$

$$\dfrac{\dfrac{\Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, A^\perp \,\&\, B^\perp \vdash}\&_1 \qquad \dfrac{\Delta, A \vdash \mathrm{b} \qquad \Delta, B \vdash \mathrm{c}}{\Delta, A \oplus B \vdash}\oplus}{\Gamma, \Delta \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \Delta, A \vdash \mathrm{b}}{\Gamma, \Delta \vdash}\textsc{Fuse}$$

$$\dfrac{\dfrac{\Gamma, B^\perp \vdash \mathrm{a}}{\Gamma, A^\perp \,\&\, B^\perp \vdash}\&_2 \qquad \dfrac{\Delta, A \vdash \mathrm{b} \qquad \Delta, B \vdash \mathrm{c}}{\Delta, A \oplus B \vdash}\oplus}{\Gamma, \Delta \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\Gamma, B^\perp \vdash \mathrm{a} \qquad \Delta, B \vdash \mathrm{c}}{\Gamma, \Delta \vdash}\textsc{Fuse}$$

$$\dfrac{\dfrac{}{\bot \vdash}\bot \qquad \dfrac{\Gamma \vdash \mathrm{a}}{\Gamma, 1 \vdash}1}{\Gamma \vdash}\textsc{Fuse} \Longrightarrow \Gamma \vdash \mathrm{a}$$

$$\dfrac{\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \Delta, A^\perp \vdash \mathrm{c}}{\Gamma^n, \Delta^m, \bigparr_{n+m} A^\perp \vdash}\parr \qquad \dfrac{\Xi, A^{n+m} \vdash \mathrm{b}}{\Xi, \bigotimes_{n+m} A \vdash}\bigotimes}{\Gamma^n, \Delta^m, \Xi \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A^\perp \vdash \mathrm{c} \qquad \dfrac{\Xi, A^{n+m} \vdash \mathrm{b}}{\Xi, A^n, A^m \vdash}\textsc{Merge}_n}{\Delta^m, \Xi, A^n \vdash}\textsc{Fuse}_m}{\Gamma^n, \Delta^m, \Xi \vdash}\textsc{Fuse}_n$$

$$\dfrac{\dfrac{\Gamma, A^\perp \vdash \mathrm{a}}{\Gamma^{n+m}, \S_{n+m} A^\perp \vdash}\S \qquad \dfrac{\Delta, A \vdash \mathrm{b} \qquad \Xi, A \vdash \mathrm{c}}{\Delta^n, \Xi^m, \S_{n+m} A \vdash}\S}{\Gamma^{n+m}, \Delta^n, \Xi^m \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\dfrac{\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \Delta, A \vdash \mathrm{b}}{\Delta, \Gamma \vdash}\textsc{Fuse} \qquad \dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \Xi, A \vdash \mathrm{c}}{\Xi, \Gamma \vdash}\textsc{Fuse}}{\Delta^n, \Xi^m, \Gamma^n, \Gamma^m \vdash}\S}{\Gamma^{n+m}, \Delta^n, \Xi^m \vdash}\textsc{Split}_n$$

Structural reduction rules.

$$\dfrac{\dfrac{\Gamma, A^{\perp m} \vdash \mathrm{a} \qquad \Delta, B^{\perp n}, A \vdash \mathrm{b}}{\Gamma, \Delta^m, B^{\perp nm} \vdash}\textsc{Cut}_m \qquad \Xi, B \vdash \mathrm{c}}{\Gamma, \Delta^m, \Xi^{nm} \vdash}\textsc{Fuse}_{nm} \Longrightarrow \dfrac{\Gamma, A^{\perp m} \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B^{\perp n} \vdash \mathrm{b} \qquad \Xi, B \vdash \mathrm{c}}{\Delta, \Xi^n, A \vdash}\textsc{Fuse}_n}{\Gamma, \Delta^m, \Xi^{nm} \vdash}\textsc{Cut}_m$$

$$\dfrac{\dfrac{\Delta, A^\perp \vdash \mathrm{a} \qquad \Gamma, B^\perp, A^n \vdash \mathrm{b}}{\Gamma, \Delta^n, B^\perp \vdash}\textsc{Cut}_n \qquad \Xi, B^m \vdash \mathrm{c}}{\Gamma^m, \Delta^{nm}, \Xi \vdash}\textsc{Fuse}_m \Longrightarrow \dfrac{\Delta, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Gamma, A^n, B^\perp \vdash \mathrm{b} \qquad \Xi, B^m \vdash \mathrm{c}}{\Gamma^m, \Xi, A^{nm} \vdash}\textsc{Fuse}_m}{\Gamma^m, \Delta^{nm}, \Xi \vdash}\textsc{Cut}_{nm}$$

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a}[x] \qquad \dfrac{}{A^\perp, A \vdash}\textsc{Ax}}{\Gamma, A^\perp \vdash}\textsc{Fuse} \Longrightarrow \Gamma, A^\perp \vdash \mathrm{a}[w]$$

$$\dfrac{\Delta, A^\perp \vdash \mathrm{b} \qquad \dfrac{\Gamma, A^m, A^n \vdash \mathrm{a}}{\Gamma, A^{n+m} \vdash}\textsc{Split}_m}{\Gamma, \Delta^{n+m} \vdash}\textsc{Fuse}_{n+m} \Longrightarrow \dfrac{\dfrac{\Delta, A^\perp \vdash \mathrm{b} \qquad \dfrac{\Delta, A^\perp \vdash \mathrm{b} \qquad \Gamma, A^m, A^n \vdash \mathrm{a}}{\Gamma, \Delta^n, A^m \vdash}\textsc{Fuse}_n}{\Gamma, \Delta^m, \Delta^n \vdash}\textsc{Fuse}_m}{\Gamma, \Delta^{n+m} \vdash}\textsc{Split}_m$$

Commuting conversions

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B, C \vdash \mathrm{b}}{B \otimes C, \Delta, A \vdash}\otimes}{\Gamma, B \otimes C, \Delta \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\dfrac{\Delta, B, C, A \vdash \mathrm{b} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, B, C \vdash}\textsc{Fuse}}{\Gamma, B \otimes C, \Delta \vdash}\otimes$$

$$\dfrac{\Xi, C^\perp \vdash \mathrm{a} \qquad \dfrac{\Gamma, A \vdash \mathrm{b} \qquad \Delta, C, B \vdash \mathrm{c}}{\Gamma, A \parr B, \Delta, C \vdash}\parr}{\Gamma, A \parr B, \Delta, \Xi \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\Gamma, A \vdash \mathrm{b} \qquad \dfrac{\Delta, B, C \vdash \mathrm{c} \qquad \Xi, C^\perp \vdash \mathrm{a}}{\Delta, \Xi, B \vdash}\textsc{Fuse}}{\Gamma, A \parr B, \Delta, \Xi \vdash}\parr$$

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B \vdash \mathrm{b} \qquad \Delta, A, C \vdash \mathrm{c}}{B \oplus C, \Delta, A \vdash}\oplus}{\Gamma, B \oplus C, \Delta \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\dfrac{\Delta, B, A \vdash \mathrm{b} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, B \vdash}\textsc{Fuse} \qquad \dfrac{\Delta, C, A \vdash \mathrm{c} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, C \vdash}\textsc{Fuse}}{\Gamma, B \oplus C, \Delta \vdash}\oplus$$

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A \vdash \mathrm{b} \qquad \Delta, A \vdash \mathrm{c}}{\Delta, A \vdash}\textsc{compare}}{\Gamma, \Delta \vdash}\textsc{Fuse} \Longrightarrow \dfrac{\dfrac{\Delta, A \vdash \mathrm{b} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta \vdash}\textsc{Fuse} \qquad \dfrac{\Delta, A \vdash \mathrm{c} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta \vdash}\textsc{Fuse}}{\Gamma, \Delta \vdash}\textsc{compare}$$

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A^n, B^n \vdash \mathrm{b}}{\bigotimes_n B, \Delta, A^n \vdash}\bigotimes}{\Gamma^n, \bigotimes_n B, \Delta \vdash}\textsc{Fuse}_n \Longrightarrow \dfrac{\dfrac{\Delta, B^n, A^n \vdash \mathrm{b} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma^n, \Delta, B^n \vdash}\textsc{Fuse}_n}{\Gamma^n, \bigotimes_n B, \Delta \vdash}\bigotimes$$

$$\dfrac{\Gamma, A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B \vdash \mathrm{b} \qquad \Xi, B \vdash \mathrm{c}}{\Xi^m, \bigparr_{n+m} B, \Delta^n, A^n \vdash}\parr}{\Gamma^n, \Xi^m, \bigparr_{n+m} B, \Delta^n \vdash}\textsc{Fuse}_n \Longrightarrow \dfrac{\dfrac{\Delta, B, A \vdash \mathrm{b} \qquad \Gamma, A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, B \vdash}\textsc{Fuse} \qquad \Xi, B \vdash \mathrm{c}}{\Gamma^n, \Xi^m, \bigparr_{n+m} B, \Delta^n \vdash}\parr$$

$$\dfrac{\Gamma, A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B \vdash \mathrm{b} \qquad \Xi, B \vdash \mathrm{c}}{\Xi^m, \S_{n+m} B, \Delta^n, A^n \vdash}\,\S}{\Gamma^n, \Xi^m, \S_{n+m} B, \Delta^n \vdash}\,\textsc{Fuse}_n \Longrightarrow \dfrac{\dfrac{\Delta, B, A \vdash \mathrm{b} \qquad \Gamma, A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta, B \vdash}\,\textsc{Fuse} \qquad \Xi, B \vdash \mathrm{c}}{\Gamma^n, \Xi^m, \S_{n+m} B, \Delta^n \vdash}\,\S$$

$$\dfrac{\Gamma, A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, B^{\perp m} \vdash \mathrm{b} \qquad \Xi, A^n, B^{mk} \vdash \mathrm{c}}{\Delta, \Xi, A^n \vdash}\,\textsc{sync}^k_m}{\Gamma^n, \Delta, \Xi \vdash}\,\textsc{Fuse}_n \Longrightarrow \dfrac{\Delta, B^{\perp m} \vdash \mathrm{b} \qquad \dfrac{\Xi, B^{mk}, A^n \vdash \mathrm{c} \qquad \Gamma, A^{\perp} \vdash \mathrm{a}}{\Gamma^n, \Xi, B^{mk} \vdash}\,\textsc{Fuse}_n}{\Gamma^n, \Delta, \Xi \vdash}\,\textsc{sync}^k_m$$

$$\dfrac{\Gamma, A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, A^{\perp} \vdash \mathrm{b} \qquad \Xi, A^n, \S_m(A \otimes (A^{\perp} \,\&\, 1)) \vdash \mathrm{c}}{\Delta, \Xi, A^n \vdash}\,\textsc{loop}}{\Gamma^n, \Delta, \Xi \vdash}\,\textsc{Fuse}_n \Longrightarrow \dfrac{\Delta, A^{\perp} \vdash \mathrm{b} \qquad \dfrac{\Xi, \S_m(A \otimes (A^{\perp} \,\&\, 1)), A^n \vdash \mathrm{c} \qquad \Gamma, A^{\perp} \vdash \mathrm{a}}{\Gamma^n, \Xi, \S_m(A \otimes (A^{\perp} \,\&\, 1)) \vdash}\,\textsc{Fuse}_n}{\Gamma^n, \Delta, \Xi \vdash}\,\textsc{loop}$$

$$\dfrac{\dfrac{\Gamma, A^m, A^n \vdash \mathrm{a}}{\Gamma, A^{n+m} \vdash}\,\textsc{Split}_m \qquad \Delta, A^{\perp} \vdash \mathrm{b}}{\Gamma, \Delta^{n+m} \vdash}\,\textsc{Fuse}_{n+m} \Longrightarrow \dfrac{\dfrac{\Delta, A^{\perp} \vdash \mathrm{b} \qquad \dfrac{\Delta, A^{\perp} \vdash \mathrm{b} \qquad \Gamma, A^m, A^n \vdash \mathrm{a}}{\Gamma, \Delta^n, A^m \vdash}\,\textsc{Fuse}_n}{\Gamma, \Delta^m, \Delta^n \vdash}\,\textsc{Fuse}_m}{\Gamma, \Delta^{n+m} \vdash}\,\textsc{Split}_m$$

$$\dfrac{\Gamma, A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, A, B \vdash \mathrm{b}}{B \,\&\, C, \Delta, A \vdash}\,\&_1}{\Gamma, B \,\&\, C, \Delta \vdash}\,\textsc{Fuse} \Longrightarrow \dfrac{\dfrac{\Delta, B, A \vdash \mathrm{b} \qquad \Gamma, A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta, B \vdash}\,\textsc{Fuse}}{\Gamma, B \,\&\, C, \Delta \vdash}\,\&_1$$

Merge reduction: base cases

$$\dfrac{\dfrac{\Delta, A, B \vdash \mathrm{a}}{\S_{m+n} B, \Delta^{m+n}, A^{m+n} \vdash}\,\S}{\S_{m+n} B, A^m, A^n, \Delta^{m+n} \vdash}\,\textsc{Merge}_m \Longrightarrow \dfrac{\dfrac{A, \Delta, B \vdash \mathrm{a} \qquad A, \Delta, B \vdash \mathrm{a}}{\S_{m+n} B, A^m, A^n, \Delta^m, \Delta^n \vdash}\,\S}{\S_{m+n} B, A^m, A^n, \Delta^{m+n} \vdash}\,\textsc{Split}_m$$

$$\dfrac{\dfrac{\Delta, A, B \vdash \mathrm{a}}{⅋_{m+n} B, \Delta^{m+n}, A^{m+n} \vdash}\,⅋}{⅋_{m+n} B, A^m, A^n, \Delta^{m+n} \vdash}\,\textsc{Merge}_m \Longrightarrow \dfrac{\dfrac{A, \Delta, B \vdash \mathrm{a} \qquad A, \Delta, B \vdash \mathrm{a}}{⅋_{m+n} B, A^m, A^n, \Delta^m, \Delta^n \vdash}\,⅋}{⅋_{m+n} B, A^m, A^n, \Delta^{m+n} \vdash}\,\textsc{Split}_m$$

$$\dfrac{\dfrac{\Gamma, A, B^{\perp} \vdash \mathrm{a} \qquad \Delta, B^{m+n} \vdash \mathrm{b}}{\Gamma^{m+n}, \Delta, A^{m+n} \vdash}\,\textsc{Cut}_{m+n}}{\Gamma^{m+n}, \Delta, A^m, A^n \vdash}\,\textsc{Merge}_m \Longrightarrow \dfrac{\dfrac{A, \Gamma, B^{\perp} \vdash \mathrm{a} \qquad \dfrac{A, \Gamma, B^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, B^{m+n} \vdash \mathrm{b}}{\Delta, B^m, B^n \vdash}\,\textsc{Merge}_m}{\Delta, A^n, \Gamma^n, B^m \vdash}\,\textsc{Cut}_n}{\Delta, A^m, A^n, \Gamma^m, \Gamma^n \vdash}\,\textsc{Cut}_m}{\Gamma^{m+n}, \Delta, A^m, A^n \vdash}\,\textsc{Split}_m$$

We list reduction rules operating with one given orientation of the cut. For each reduction, the variant resulting from its composition with the cut-swap must be considered.

$$\dfrac{\Gamma, A^{\perp n} \vdash \mathrm{a} \qquad \Delta, A \vdash \mathrm{b}}{\Gamma, \Delta^n \vdash}\,\textsc{Fuse}_n \Longrightarrow \dfrac{\Delta, A \vdash \mathrm{b} \qquad \Gamma, A^{\perp n} \vdash \mathrm{a}}{\Gamma, \Delta^n \vdash}\,\textsc{Fuse}_n$$

2. Semantics as λ-terms

These operations are pervasive in the semantics:

- Shifting control: In a computation that produces $\perp$ as a result, we obtain a value of type α from a value of type $(\alpha \to \perp) \to \perp$ by passing a continuation of type $\alpha \to \perp$.
- Indexing into n-ary variables, that is, converting from $\mathbb{N} \times (\mathbb{N} \to \alpha)$ to α by evaluating the function (second projection) on the offset (first projection).
- Introducing 1-ary variables, that is, converting from α to $\mathbb{N} \times (\mathbb{N} \to \alpha)$ by going from x to $(0,\ \lambda\!\!\!-\!.\ x)$.
- Merging two variables into one, such that $\mathrm{merge}\, n\, (s,\ f)\, (t,\ g) \equiv (0,\ \lambda i.\ \mathrm{if}\, i\ \leq\ n\, \mathrm{then}\, f\, (s+i)\, \mathrm{else}\, g\, (t+i-n))$.

$$\left(\dfrac{}{x : \mathbb{A}, y : \mathbb{A}^{\perp} \vdash x \leftrightarrow y}\,\mathrm{Ax} \right)^{\bullet} =$$
$\quad \mathrm{let}\ s,\ \mu\ =\ x\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0\ =\ y\ \mathrm{in}\ \mu\ s\ (\mu_0\ s_0)$

$$\left(\dfrac{\Gamma, x : A \vdash \mathrm{a} \qquad \Delta, y : A^{\perp} \vdash \mathrm{b}}{\Gamma, \Delta \vdash \mathrm{cut}\{x : A \mapsto \mathrm{a};\, y : A^{\perp} \mapsto \mathrm{b}\}}\,\textsc{Cut} \right)^{\bullet} =$$
$\quad \mathrm{let}\ x\ =\ (0,\ \lambda y.\ \mathrm{let}\ y_0\ =\ (0,\ \lambda\_.\ y)\ \mathrm{in}\ b^{\bullet}\,[\Delta, y_0])\ \mathrm{in}$
$\quad a^{\bullet}\,[\Gamma, x]$

$$\left(\dfrac{\Gamma, x : P^{\perp} \vdash \mathrm{a} \qquad \Delta, y : P^n \vdash \mathrm{b}}{\Gamma^n, \Delta \vdash \mathrm{cut}\{x : P^{\perp} \mapsto \mathrm{a};\, y : P^n \mapsto \mathrm{b}\}}\,\textsc{Cut}_n \right)^{\bullet} =$$

let $y = (0,\ \lambda i.\ \lambda x.$ let $s,\ \mu = \Gamma$ in $(\lambda \gamma_0.$
$a^\bullet[\gamma_0,x])\ (i{+}s,\ \mu))$ in $b^\bullet[\Delta,y]$

$$\left(\frac{\Gamma, x : N^\perp \vdash \mathrm{a} \qquad \Delta, y : N^n \vdash \mathrm{b}}{\Gamma^n, \Delta \vdash \mathrm{cut}\{x : N^\perp \mapsto \mathrm{a}; y : N^n \mapsto \mathrm{b}\}}\ \textsc{Cut}_n\right)^\bullet =$$

let $y = (0,\ \lambda i.\ \lambda x.$ let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in
let $s,\ \mu = \Gamma$ in $(\lambda\gamma_0.\ a^\bullet[\gamma_0,x_0])\ (i{+}s,\ \mu))$ in $b^\bullet[\Delta,y]$

$$\left(\frac{\Gamma \vdash \mathrm{a} \qquad \Delta \vdash \mathrm{b}}{\Gamma, \Delta \vdash \mathrm{mix}\{\mathrm{a}; \mathrm{b}\}}\ \textsc{Mix}\right)^\bullet =$$

$a^\bullet[\Gamma] \gg b^\bullet[\Delta]$

$$\left(\frac{}{x : \curlywedge \vdash \text{yield to } x}\ \curlywedge\right)^\bullet =$$

let $s,\ \mu = x$ in $\mu\ s$

$$\left(\frac{\Gamma \vdash \mathrm{a}}{\Gamma, x : 1 \vdash \text{let } \diamond = x; \mathrm{a}}\ 1\right)^\bullet =$$

let $s,\ \mu = x$ in $\mu\ s\ a^\bullet[\Gamma]$

$$\left(\frac{}{\vdash \mathrm{halt}}\ \textsc{Halt}\right)^\bullet = \mathrm{nop}$$

$$\left(\frac{}{\Gamma, x : 0 \vdash \text{dump } \Gamma \text{ in } x}\ 0\right)^\bullet =$$

let $s,\ \mu = x$ in $\mu\ s$

$$\left(\frac{\Gamma, x : A, y : B \vdash \mathrm{a}}{\Gamma, z : A \otimes B \vdash \text{let } x, y = z; \mathrm{a}}\ \otimes\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s\ (\lambda(x,y) \mapsto$let $y_0 = (0,\ \lambda\_.\ y)$ in
let $x_0 = (0,\ \lambda\_.\ x)$ in $a^\bullet[\Gamma,x_0,y_0])$

$$\left(\frac{\Gamma, x : A \vdash \mathrm{a} \qquad \Delta, y : B \vdash \mathrm{b}}{\Gamma, z : A \parr B, \Delta \vdash \text{connect } z \text{ to}\{x \mapsto \mathrm{a}; y \mapsto \mathrm{b}\}}\ \parr\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s\ (\lambda x.$
let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in $a^\bullet[\Gamma,x_0],\ \lambda y.$
let $y_0 = (0,\ \lambda\_.\ \lambda\kappa_0.\ \kappa_0 \propto y)$ in $b^\bullet[\Delta,y_0])$

$$\left(\frac{\Gamma, x : A \vdash \mathrm{a} \qquad \Gamma, y : B \vdash \mathrm{b}}{\Gamma, z : A \oplus B \vdash \text{case } z \text{ of}\{\text{inl } x \mapsto \mathrm{a}; \text{inr } y \mapsto \mathrm{b}\}}\ \oplus\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s\ \lambda($inl $x \mapsto$ let $x_0 = (0,\ \lambda\_.\ x)$ in
$a^\bullet[\Gamma,x_0]$ |inr $y \mapsto$ let $y_0 = (0,\ \lambda\_.\ y)$ in $b^\bullet[\Gamma,y_0]$)

$$\left(\frac{\Gamma, x : A \vdash \mathrm{a}}{\Gamma, z : A \& B \vdash \text{let inl } x = z; \mathrm{a}}\ \&_1\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s$ (inl $(\lambda x.$
let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in $a^\bullet[\Gamma,x_0]))$

$$\left(\frac{\Gamma, x : B \vdash \mathrm{a}}{\Gamma, z : A \& B \vdash \text{let inr } x = z; \mathrm{a}}\ \&_2\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s$ (inr (λx.
let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in $a^\bullet[\Gamma,x_0]$))

$$\left(\frac{\Gamma, x : A^n, y : A^m \vdash \mathrm{a}}{\Gamma, z : A^{n+m} \vdash \text{let } x, y = \text{split}_n\ z; \mathrm{a}}\ \textsc{Split}_n \right)^\bullet =$$

let $n_0,\ z_0 = z$ in let $x = (n,\ z_0)$ in let $y = (m,\ z_0)$ in
$a^\bullet[\Gamma,x,y]$

$$\left(\frac{\Gamma, x : A^m \vdash \mathrm{a}}{\Gamma, z : \bigotimes_m A \vdash \text{let } x = \text{slice } z; \mathrm{a}}\ \bigotimes \right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s$ (λx. let $x_0 = (m,\ x)$ in $a^\bullet[\Gamma,x_0]$)

$$\left(\frac{\Gamma, x : A \vdash \mathrm{a} \qquad \Delta, y : A \vdash \mathrm{b}}{\Gamma^n, \Delta^m, z : \bigparr_{n+m} A \vdash \text{coslice } z\{x \mapsto_n \mathrm{a}; y \mapsto_m \mathrm{b}\}}\ \bigparr \right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s$ (λi.
if $0 \le i \wedge i < n$ then λx.
let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in let $s_0,\ \mu_0 = \Gamma$ in
$(\lambda\gamma_0.\ a^\bullet[\gamma_0,x_0])\ (i{+}s_0,\ \mu_0)$
else λy. let $y_0 = (0,\ \lambda\_.\ \lambda\kappa_0.\ \kappa_0 \propto y)$ in
let $s_1,\ \mu_1 = \Delta$ in $(\lambda\delta_0.\ b^\bullet[\delta_0,y_0])\ ({-n}{+}i{+}s_1,\ \mu_1)$)

$$\left(\frac{\Gamma, x' : A, y' : B \vdash \mathrm{a}}{\Gamma^n, x : \S^+_n A, y : \S^+_n B \vdash \text{traverse}\{x \text{ as } x', y \text{ as } y' \mapsto_n \mathrm{a}\}}\ \S \right)^\bullet =$$

let $s,\ \mu = x$ in let $s_0,\ \mu_0 = y$ in schedule (λi.
$\mu\ s\ i$ ($\lambda x'$. let $x'_0 = (0,\ \lambda\_.\ x')$ in $\mu_0\ s_0\ i$ ($\lambda y'$.
let $y'_0 = (0,\ \lambda\_.\ y')$ in let $s_1,\ \mu_1 = \Gamma$ in ($\lambda\gamma_0$.
$a^\bullet[x'_0,y'_0,\gamma_0]$) $(i{+}s_1,\ \mu_1)$)))

$$\left(\frac{\Gamma, x' : A, y' : B \vdash \mathrm{a}}{\Gamma^n, x : \S^-_n A, y : \S^+_n B \vdash \text{traverse}\{x \text{ as } x', y \text{ as } y' \mapsto_n \mathrm{a}\}}\ \S \right)^\bullet =$$

let $s,\ \mu = y$ in let $s_0,\ \mu_0 = x$ in $\mu_0\ s_0$ ($\lambda i.\ \lambda x'$.
let $x'_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x')$ in $\mu\ s\ i$ ($\lambda y'$.
let $y'_0 = (0,\ \lambda\_.\ y')$ in let $s_1,\ \mu_1 = \Gamma$ in ($\lambda\gamma_0$.
$a^\bullet[x'_0,y'_0,\gamma_0]$) $(i{+}s_1,\ \mu_1)$))

$$\left(\frac{\Gamma, x : D^{\perp n} \vdash \mathrm{a} \qquad \Delta, y : D^n \vdash \mathrm{b}}{\Gamma, \Delta \vdash \text{sync}\{x : D^{\perp n} \mapsto \mathrm{a}; y : D^n \mapsto \mathrm{b}\}}\ \textsc{sync}_n \right)^\bullet =$$

allocate (λd_0. … (allocate (λd_{n-1}.
(let $x = (0,\ \lambda\_.\ (n,\ \lambda i.$ write d_i)) in $a^\bullet[\Gamma,x]$) $\gg$
(let $y = (n,\ \lambda i_0.$ read d_i) in $b^\bullet[\Delta,y]$)))))

$$\left(\frac{\Gamma, x : D^\perp \vdash \mathrm{a} \qquad \Delta, y : \S_m(D \otimes (D^\perp \,\&\, 1)) \vdash \mathrm{b}}{\Gamma, \Delta \vdash \text{loop}\{x : D^\perp \mapsto \mathrm{a}; y : \S_m(D \otimes (D^\perp \,\&\, 1)) \mapsto \mathrm{b}\}}\ \textsc{loop} \right)^\bullet =$$

allocate (λd. (let $x = (0,\ \lambda\_.$ write d) in $a^\bullet[\Gamma,x]$) $\gg$
(let $y = (m,\ \lambda i.\ \lambda u.$ read d ($\lambda r.\ u$ ($r,\ \lambda u_0$. case u_0 of
inl $w \mapsto$ write d | inr $o \mapsto o$))) in $b^\bullet[\Delta,y]$))

$$\left(\frac{\Gamma, x : A[n] \vdash \mathrm{a}}{\Gamma, z : \forall\alpha : \mathbb{N}.A[\alpha] \vdash \text{let } x = z@n; \mathrm{a}}\ \forall \right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s\ (n,\ \lambda x.$
let $x_0 = (0,\ \lambda\_.\ \lambda\kappa.\ \kappa \propto x)$ in $a^\bullet[\Gamma, x_0])$

$$\left(\frac{\Gamma, x : A[\beta] \vdash \mathrm{a}}{\Gamma, z : \exists\alpha : \mathbb{N}.A[\alpha] \vdash \text{let } x\langle\beta\rangle = z; \mathrm{a}}\ \exists\right)^\bullet =$$

let $s,\ \mu = z$ in $\mu\ s\ (\lambda(\beta,x) \mapsto$let $x_0 = (0,\ \lambda\_.\ x)$ in $a^\bullet[\Gamma, x_0])$

$$\left(\frac{\Gamma \vdash \mathrm{a} \qquad \Gamma \vdash \mathrm{b}}{\Gamma \vdash \text{compare } n, m\{\geq\ \ \mapsto \mathrm{a}; \leq\ \ \mapsto \mathrm{b}\}}\ \textsc{compare}\right)^\bullet =$$

if $n \geq m$ then $a^\bullet[\Gamma]$ else $b^\bullet[\Gamma]$

3. Polarization

3.1 Example of Non-polarizable derivation.

The impossibility can be easily verified by enumerating all possible assignments of polarities.

Left side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1, A^\perp, B^\perp \vdash \mathrm{a}}{\Gamma_1{}^n, \S_n A^\perp, \S_n B^\perp \vdash}\ \S \qquad \dfrac{\Delta_1, C^\perp \vdash \mathrm{b}}{\Delta_1{}^n, \S_n C^\perp \vdash}\ \S}{\Gamma_1{}^n, \Delta_1{}^n, \S_n A^\perp, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \parr}{\Gamma_1{}^n, \Delta_1{}^n, \S_n A^\perp, \S_n B^\perp \parr \S_n C^\perp, 1 \vdash}\ 1 \qquad \dfrac{\dfrac{\dfrac{\Gamma_1, B^\perp \vdash \mathrm{c}}{\Gamma_1{}^n, \S_n B^\perp \vdash}\ \S \qquad \dfrac{\Delta_1, A^\perp, C^\perp \vdash \mathrm{d}}{\Delta_1{}^n, \S_n A^\perp, \S_n C^\perp \vdash}\ \S}{\Gamma_1{}^n, \Delta_1{}^n, \S_n A^\perp, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \parr}{\Gamma_1{}^n, \Delta_1{}^n, \S_n A^\perp, \S_n B^\perp \parr \S_n C^\perp, 1 \vdash}\ 1}{1 \oplus 1, \Gamma_1{}^n, \Delta_1{}^n, \S_n A^\perp, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \oplus \qquad \dfrac{\dfrac{\Xi_1, B, C \vdash \mathrm{e}}{\Xi_1{}^n, \S_n B, \S_n C \vdash}\ \S}{\Xi_1{}^n, \S_n B \otimes \S_n C \vdash}\ \otimes}{1 \oplus 1, \Gamma_1{}^n, \Delta_1{}^n, \Xi_1{}^n, \S_n A^\perp \vdash}\ \textsc{Cut}$$

Right side

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_2, A, B^\perp \vdash \mathrm{f}}{\Gamma_2{}^n, \S_n A, \S_n B^\perp \vdash}\ \S \qquad \dfrac{\Delta_2, C^\perp \vdash \mathrm{g}}{\Delta_2{}^n, \S_n C^\perp \vdash}\ \S}{\Gamma_2{}^n, \Delta_2{}^n, \S_n A, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \parr}{\Gamma_2{}^n, \Delta_2{}^n, \S_n A, \S_n B^\perp \parr \S_n C^\perp, 1 \vdash}\ 1 \qquad \dfrac{\dfrac{\dfrac{\Gamma_2, B^\perp \vdash \mathrm{h}}{\Gamma_2{}^n, \S_n B^\perp \vdash}\ \S \qquad \dfrac{\Delta_2, A, C^\perp \vdash \mathrm{i}}{\Delta_2{}^n, \S_n A, \S_n C^\perp \vdash}\ \S}{\Gamma_2{}^n, \Delta_2{}^n, \S_n A, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \parr}{\Gamma_2{}^n, \Delta_2{}^n, \S_n A, \S_n B^\perp \parr \S_n C^\perp, 1 \vdash}\ 1}{1 \oplus 1, \Gamma_2{}^n, \Delta_2{}^n, \S_n A, \S_n B^\perp \parr \S_n C^\perp \vdash}\ \oplus \qquad \dfrac{\dfrac{\Xi_2, B, C \vdash \mathrm{j}}{\Xi_2{}^n, \S_n B, \S_n C \vdash}\ \S}{\Xi_2{}^n, \S_n B \otimes \S_n C \vdash}\ \otimes}{1 \oplus 1, \Gamma_2{}^n, \Delta_2{}^n, \Xi_2{}^n, \S_n A \vdash}\ \textsc{Cut}$$

Full program

$$\frac{1 \oplus 1, \Gamma_1{}^n, \Delta_1{}^n, \Xi_1{}^n, \S_n A^\perp \vdash \mathrm{a} \qquad 1 \oplus 1, \Gamma_2{}^n, \Delta_2{}^n, \Xi_2{}^n, \S_n A \vdash \mathrm{b}}{1 \oplus 1, 1 \oplus 1, \Gamma_1{}^n, \Gamma_2{}^n, \Delta_1{}^n, \Delta_2{}^n, \Xi_1{}^n, \Xi_2{}^n \vdash}\ \textsc{Cut}$$

3.2 Polarization scheme

When the translation commutes with the operator, the rule stays the same.

The polarization scheme for $\parr$ generalizes to the n-ary case. Observe how the invariant of *at most one negative name in the context* is preserved across all branches.

$$\left(\frac{\Gamma, w : A \vdash \mathrm{a} \qquad g : X, x : A \vdash \mathrm{b} \qquad \Delta, y : A \vdash \mathrm{c}}{\Gamma^n, g : X, z : \parr_{n+m+1} A, \Delta^m \vdash}\ \parr\right)^-_g = \dfrac{\dfrac{\dfrac{\dfrac{\Gamma^+, w^- : A^- \vdash \mathrm{a}}{\Gamma^{+n}, z : \parr_n A^- \vdash}\ \parr \qquad \dfrac{g^- : X^-, x^+ : A^+ \vdash \mathrm{b} \qquad \dfrac{\Delta^+, y^- : A^- \vdash \mathrm{c}}{\Delta^{+m}, z : \parr_m A^- \vdash}\ \parr}{g^- : X^-, \Delta^{+m}, z : A^+ \parr \parr_m A^- \vdash}\ \parr}{\Gamma^{+n}, g^- : X^-, \Delta^{+m}, z : \parr_n A^- \parr A^+ \parr \parr_m A^- \vdash}\ \parr}{\Gamma^{+n}, g^- : X^-, \Delta^{+m}, z : \forall i : \mathbb{N}. \parr_i A^- \parr A^+ \parr \parr_{n+m-i} A^- \vdash}\ \forall}{\Gamma^{+n}, g^- : X^-, z^+ : \parr_{n+m+1} A^- \,\&\, (\forall i : \mathbb{N}. \parr_i A^- \parr A^+ \parr \parr_{n+m-i} A^-), \Delta^{+m} \vdash}\ \&_2$$

$$\left(\frac{\Gamma, x : A \vdash \mathrm{a} \qquad \Delta, y : A \vdash \mathrm{b}}{\Gamma^n, \Delta^m, z : \parr_{n+m} A \vdash}\ \parr\right)^+ = \dfrac{\dfrac{\Gamma^+, x^- : A^- \vdash \mathrm{a} \qquad \Delta^+, y^- : A^- \vdash \mathrm{b}}{\Gamma^{+n}, \Delta^{+m}, z : \parr_{n+m} A^- \vdash}\ \parr}{\Gamma^{+n}, \Delta^{+m}, z^+ : \parr_{n+m} A^- \,\&\, (\forall i : \mathbb{N}. \parr_i A^- \parr A^+ \parr \parr_{n+m-i-1} A^-) \vdash}\ \&_1$$

The scheme for $\bigotimes_n A$ is a straight dualization of the scheme for $\bigparr_n A$.

Finally, when a TRAVERSE rule is applied, at most one sequence will be negative:

$$(\frac{\Gamma, y : C, x : A \vdash \mathrm{a} \qquad \Delta, y : C, x : B \vdash \mathrm{b}}{\Gamma^n, \Delta^m, x_1 : \S_n A, x_2 : \S_m B, y_1 : \S_{n+m} C \vdash} \S)^-_{x_1} = \frac{\Gamma^+, y^+ : C^+, x^- : A^- \vdash \mathrm{a} \qquad \Delta^+, y^+ : C^+, x^+ : B^+ \vdash \mathrm{b}}{\Gamma^{+n}, \Delta^{+m}, x_1^- : \S^-{}_n A^-, x_2^+ : \S^+{}_m B^+, y_1^+ : \S^+{}_{n+m} C^+ \vdash} \S$$

$$(\frac{\Gamma, y : C, x : A \vdash \mathrm{a} \qquad \Delta, y : C, x : B \vdash \mathrm{b}}{\Gamma^n, \Delta^m, x_1 : \S_n A, x_2 : \S_m B, y_1 : \S_{n+m} C \vdash} \S)^+ = \frac{\Gamma^+, y^+ : C^+, x^+ : A^+ \vdash \mathrm{a} \qquad \Delta^+, y^+ : C^+, x^+ : B^+ \vdash \mathrm{b}}{\Gamma^{+n}, \Delta^{+m}, x_1^+ : \S^+{}_n A^+, x_2^+ : \S^+{}_m B^+, y_1^+ : \S^+{}_{n+m} C^+ \vdash} \S$$

4. Measure

4.1 Cost of programs

Measure of sequential program execution cost in CLL_p^n. For a fixed environment, the cost is derived from the number of β-reductions and pattern matchings required to reduce the semantics of the program to a normal form. Note that let statements should be read as a syntactic substitution, and thus lack run-time cost.

- $\chi_+(A)$ is 1 if A is a positive type; 0 otherwise. It is used to account for the cost of introducing a double negation (i.e., from x to $\lambda\kappa.\kappa x$).
- $\#\mathrm{v}(\Gamma)$ is the number of names in context Γ. It is used to account for the cost of reindexing the names in a context when their arity changes (e.g. on the coslice rule).
- if and match are resolved according to an oracle which depends exclusively on the environment.

$$
\begin{aligned}
|x \leftrightarrow y|[x, y] &= 1 \\
|\mathrm{cut}\{x : A \mapsto \mathrm{a}; y : A^\perp \mapsto \mathrm{b}\}|[\Gamma, \Delta] &= |\mathrm{b}|[\Delta, y] + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{cut}\{x : A^n \mapsto \mathrm{a}; y : A^\perp \mapsto \mathrm{b}\}|[\Gamma, \Delta] &= \textstyle\sum_n (\chi_+(A^\perp) + 3 \cdot \#\mathrm{v}(\Delta) + |\mathrm{b}|[\Delta, y]) + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{mix}\{\mathrm{a}; \mathrm{b}\}|[\Gamma, \Delta] &= 2 + |\mathrm{a}|[\Gamma] + |\mathrm{b}|[\Delta] \\
|\mathrm{yield\ to}\ x|[x] &= 0 \\
|\mathrm{let}\ \diamond = x; \mathrm{a}|[\Gamma, x] &= 2 + |\mathrm{a}|[\Gamma] \\
|\mathrm{halt}|[] &= 0 \\
|\mathrm{dump}\ \Gamma\ \mathrm{in}\ x|[\Gamma, x] &= 1 \\
|\mathrm{let}\ x, y = z; \mathrm{a}|[\Gamma, z] &= 3 + |\mathrm{a}|[\Gamma, x, y] \\
|\mathrm{connect}\ z\ \mathrm{to}\{x \mapsto \mathrm{a}; y \mapsto \mathrm{b}\}|[\Gamma, z, \Delta] &= 3 + \chi_+(A) + |\mathrm{a}|[\Gamma, x] + \chi_+(B) + |\mathrm{b}|[\Delta, y] \\
|\mathrm{case}\ z\ \mathrm{of}\{\mathrm{inl}\ x \mapsto \mathrm{a}; \mathrm{inr}\ y \mapsto \mathrm{b}\}|[\Gamma, z] &= 3 + \mathrm{match}\ z\,\{\mathrm{inl}\ x \mapsto |\mathrm{a}|[\Gamma, x]\,;\ \mathrm{inr}\ y \mapsto |\mathrm{b}|[\Gamma, y]\} \\
|\mathrm{let\ inl}\ x = z; \mathrm{a}|[\Gamma, z] &= 2 + \chi_+(A) + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{let\ inr}\ x = z; \mathrm{a}|[\Gamma, z] &= 2 + \chi_+(B) + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{let}\ x, y = \mathrm{split}_n\ z; \mathrm{a}|[\Gamma, z] &= |\mathrm{a}|[\Gamma, x, y] \\
|\mathrm{let}\ x = \mathrm{slice}\ z; \mathrm{a}|[\Gamma, z] &= 2 + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{coslice}\ z\{x \mapsto_n \mathrm{a}; y \mapsto_m \mathrm{b}\}|[\Gamma, \Delta, z] &= 2 + \textstyle\sum_n (1 + 3 \cdot \#\mathrm{v}(\Gamma) + \chi_+(A) + |\mathrm{a}|[\Gamma, x]) + \sum_m (1 + 3 \cdot \#\mathrm{v}(\Delta) + \chi_+(A) + |\mathrm{b}|[\Delta, y]) \\
|\mathrm{traverse}\{x : \S^+{}_n A\ \mathrm{as}\ x', y : \S^+{}_n B\ \mathrm{as}\ y' \mapsto_n \mathrm{a}\}|[\Gamma, x, y] &= 4 + 1 + \textstyle\sum_n (2 + 3 \cdot \#\mathrm{v}(\Gamma) + |\mathrm{a}|[\Gamma, x', y']) \\
|\mathrm{traverse}\{x : \S^-{}_n A\ \mathrm{as}\ x', y : \S^+{}_n B\ \mathrm{as}\ y' \mapsto_n \mathrm{a}\}|[\Gamma, x, y] &= 3 + \chi_+(A) + \textstyle\sum_n (1 + 3 \cdot \#\mathrm{v}(\Gamma) + |\mathrm{a}|[\Gamma, x', y']) \\
|\mathrm{sync}\{x : D^{\perp n} \mapsto \mathrm{a}; y : D^n \mapsto \mathrm{b}\}|[\Gamma, \Delta] &= |\mathrm{b}|[\Delta, y] + |\mathrm{a}|[\Gamma, x] + 9n \\
|\mathrm{loop}\{x : D^\perp \mapsto \mathrm{a}; y : \S_m (D \otimes (D^\perp \,\&\, 1)) \mapsto \mathrm{b}\}|[\Gamma, \Delta] &= |\mathrm{b}|[\Delta, y] + |\mathrm{a}|[\Gamma, x] + 7m + 6 \\
|\mathrm{let}\ x = z@n; \mathrm{a}|[\Gamma, z] &= 2 + \chi_+(A[n]) + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{let}\ x\langle\beta\rangle = z; \mathrm{a}|[\Gamma, z] &= 3 + |\mathrm{a}|[\Gamma, x] \\
|\mathrm{compare}\ n, m\{\geq\ \mapsto \mathrm{a}; \leq\ \mapsto \mathrm{b}\}|[\Gamma] &= \mathrm{if}\,\{n \geq m \mapsto |\mathrm{a}|[\Gamma]; n \leq m \mapsto |\mathrm{b}|[\Gamma]\}
\end{aligned}
$$

4.2 Proof that fusion improves the cost: cases

Main reduction rules. Note that no n-ary cut needs to be handled: the eliminators can only handle one value at a time.

$\boxed{\bigparr\otimes\mathrm{R}}$

$$|\mathrm{fuse}\{\bar{z} : A^\perp \bigparr B^\perp \mapsto \mathrm{connect}\ \bar{z}\ \mathrm{to}\{\bar{x} \mapsto \mathrm{a}; \bar{y} \mapsto \mathrm{b}\}; z : A \otimes B \mapsto \mathrm{let}\ x, y = z; \mathrm{c}\}|[\Gamma, \Delta, \xi] = 3 + \chi_+(A^\perp) + |\mathrm{a}|[\Gamma, \bar{x}] + \chi_+(B^\perp) +$$

$$|\mathrm{b}|[\Delta, \bar{y}] + 3 + |\mathrm{c}|[\xi, x, y] \geq |\mathrm{a}|[\Gamma, \bar{x}] + |\mathrm{b}|[\Delta, \bar{y}] + |\mathrm{c}|[\xi, x, y] = |\mathrm{fuse}\{\bar{x} : A^\perp \mapsto \mathrm{a}; x : A \mapsto \mathrm{fuse}\{\bar{y} : B^\perp \mapsto \mathrm{b}; y : B \mapsto \mathrm{c}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\&\oplus 1}$

$$|\text{fuse}\{z : A^\perp \& B^\perp \mapsto \text{let inl } x = z; \text{a}; \bar{z} : A \oplus B \mapsto \text{case } \bar{z} \text{ of}\{\text{inl } \bar{x} \mapsto \text{b}; \text{inr } \bar{y} \mapsto \text{c}\}\}|[\Gamma, \Delta] =$$

$$2 + \chi_+(A^\perp) + |\text{a}|[\Gamma, x] + 3 + \text{match } \bar{z}\,\{\text{inl } \bar{x} \mapsto |\text{b}|[\Delta, \bar{x}]\,;\ \text{inr } \bar{y} \mapsto |\text{c}|[\Delta, \bar{y}]\} \geq |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, \bar{x}] = |\text{fuse}\{x : A^\perp \mapsto \text{a}; \bar{x} : A \mapsto \text{b}\}|[\Gamma, \Delta]$$

$\boxed{\&\oplus 2}$

$$|\text{fuse}\{z : A^\perp \& B^\perp \mapsto \text{let inr } x = z; \text{a}; \bar{z} : A \oplus B \mapsto \text{case } \bar{z} \text{ of}\{\text{inl } \bar{x} \mapsto \text{b}; \text{inr } \bar{y} \mapsto \text{c}\}\}|[\Gamma, \Delta] =$$

$$2 + \chi_+(B^\perp) + |\text{a}|[\Gamma, x] + 3 + \text{match } \bar{z}\,\{\text{inl } \bar{x} \mapsto |\text{b}|[\Delta, \bar{x}]\,;\ \text{inr } \bar{y} \mapsto |\text{c}|[\Delta, \bar{y}]\} \geq |\text{a}|[\Gamma, x] + |\text{c}|[\Delta, \bar{y}] = |\text{fuse}\{x : B^\perp \mapsto \text{a}; \bar{y} : B \mapsto \text{c}\}|[\Gamma, \Delta]$$

$\boxed{\perp 1}$

$$|\text{fuse}\{z : \bot \mapsto \text{yield to } z; \bar{z} : 1 \mapsto \text{let } \diamond = \bar{z}; \text{a}\}|[\Gamma] = 2 + |\text{a}|[\Gamma] \geq |\text{a}|[\Gamma] = |\text{a}|[\Gamma]$$

$\boxed{\text{n}⅋\otimes}$

$$|\text{fuse}\{z : \underset{n+m}{⅋} A^\perp \mapsto \text{coslice } z\{x \mapsto_n \text{a}; y \mapsto_m \text{c}\}; \bar{z} : \bigotimes_{n+m} A \mapsto \text{let } \bar{x} = \text{slice } \bar{z}; \text{b}\}|[\Gamma, \Delta, \xi] =$$

$$2 + \sum\nolimits_n (1 + 3 \cdot \#\text{v}(\Gamma) + \chi_+(A^\perp) + |\text{a}|[\Gamma, x]) + \sum\nolimits_m (1 + 3 \cdot \#\text{v}(\Delta) + \chi_+(A^\perp) + |\text{c}|[\Delta, y]) + 2 + |\text{b}|[\xi, \bar{x}] \geq$$

$$\sum\nolimits_n (\chi_+(A^\perp) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + \sum\nolimits_m (\chi_+(A^\perp) + 3 \cdot \#\text{v}(\Delta) + |\text{c}|[\Delta, y]) + n + m + |\text{b}|[\xi, \bar{x}] = |\text{fuse}\{x : A^\perp \mapsto \text{a}; x : A^n \mapsto$$

$$\text{fuse}\{y : A^\perp \mapsto \text{c}; y : A^m \mapsto \text{let } \bar{x} = \text{merge } x, y; \text{b}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\text{n}\S}$

$$|\text{fuse}\{x : \underset{n+m}{\S^-} A^\perp \mapsto \text{traverse}\{x : \underset{n+m}{\S^-} A^\perp \text{ as } x_1 \mapsto_{n+m} \text{a}\}; y : \underset{n+m}{\S^+} A \mapsto \text{traverse}\{y : \underset{n+m}{\S^+} A \text{ as } y_1 \mapsto_n \text{b}; y : \underset{n+m}{\S^+} A \text{ as } y_2 \mapsto$$

$$_m\text{c}\}\}|[\Gamma, \Delta, \xi] = 1 + \chi_+(A^\perp) + \sum\nolimits_{n+m} (3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x_1]) + n + m + 2 + 1 + \sum\nolimits_n (1 + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, y_1]) + 1 + \sum\nolimits_m (1 + 3 \cdot \#\text{v}(\Xi) +$$

$$|\text{c}|[\xi, y_2]) \geq n + m + 1 + \sum\nolimits_n (3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x_1] + |\text{b}|[\Delta, y_1]) + 1 + \sum\nolimits_m (3 \cdot \#\text{v}(\Xi) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x_1] + |\text{c}|[\xi, y_2]) =$$

$$|\text{let } \Gamma, \Gamma = \text{split}_n\, \Gamma; \text{traverse}\{\mapsto_n \text{fuse}\{x_1 : A^\perp \mapsto \text{a}; y_1 : A \mapsto \text{b}\}; \mapsto_m \text{fuse}\{x_1 : A^\perp \mapsto \text{a}; y_2 : A \mapsto \text{c}\}\}|[\Gamma, \Delta, \xi]$$

Structural reduction rules.

$\boxed{\text{Associativity-L}}$

$$|\text{fuse}\{y : {B^\perp}^{nm} \mapsto \text{cut}\{x : {A^\perp}^m \mapsto \text{a}; \bar{x} : A \mapsto \text{b}\}; \bar{y} : B \mapsto \text{c}\}|[\Gamma, \Delta, \xi] =$$

$$\sum\nolimits_{nm} (\chi_+(B) + 3 \cdot \#\text{v}(\Xi) + |\text{c}|[\xi, \bar{y}]) + \sum\nolimits_m (\chi_+(A) + 3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(B^\perp) + |\text{b}|[\Delta, y, \bar{x}]) + |\text{a}|[\Gamma, x] \geq$$

$$\sum\nolimits_m (\chi_+(A) + 3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(\Xi) + \sum\nolimits_n (\chi_+(B) + 3 \cdot \#\text{v}(\Xi) + |\text{c}|[\xi, \bar{y}]) + |\text{b}|[\Delta, \bar{x}, y]) + |\text{a}|[\Gamma, x] = |\text{cut}\{x : {A^\perp}^m \mapsto \text{a}; \bar{x} : A \mapsto$$

$$\text{fuse}\{y : {B^\perp}^n \mapsto \text{b}; \bar{y} : B \mapsto \text{c}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\text{Associativity-R}}$

$$|\text{fuse}\{y : B^\perp \mapsto \text{cut}\{x : A^\perp \mapsto \text{a}; \bar{x} : A^n \mapsto \text{b}\}; \bar{y} : B^m \mapsto \text{c}\}|[\Gamma, \Delta, \xi] =$$

$$\sum\nolimits_m (\chi_+(B^\perp) + 3 \cdot \#\text{v}(\Gamma) + 3 \cdot \#\text{v}(\Delta) + \sum\nolimits_n (\chi_+(A^\perp) + 3 \cdot \#\text{v}(\Delta) + |\text{a}|[\Delta, x]) + |\text{b}|[\Gamma, y, \bar{x}]) + |\text{c}|[\xi, \bar{y}] \geq$$

$$\sum\nolimits_{nm} (\chi_+(A^\perp) + 3 \cdot \#\text{v}(\Delta) + |\text{a}|[\Delta, x]) + \sum\nolimits_m (\chi_+(B^\perp) + 3 \cdot \#\text{v}(\Gamma) + 3 \cdot \#\text{v}(A) + |\text{b}|[\Gamma, \bar{x}, y]) + |\text{c}|[\xi, \bar{y}] = |\text{cut}\{x : A^\perp \mapsto \text{a}; \bar{x} :$$

$$A^{nm} \mapsto \text{fuse}\{y : B^\perp \mapsto \text{b}; \bar{y} : B^m \mapsto \text{c}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\text{Ax}}$

$$|\text{fuse}\{x : A^\perp \mapsto \text{a}[x]; y : A \mapsto y \leftrightarrow w\}|[\Gamma, w] = |\text{a}[x]|[\Gamma, x] + 1 \geq |\text{a}[w]|[\Gamma, w] = |\text{a}[w]|[\Gamma, w]$$

Split

$$|\text{fuse}\{\bar{z} : A^{\perp} \mapsto \text{b}; z : A^{n+m} \mapsto \text{let } x, y = \text{split}_m\ z; \text{a}\}|[\Gamma, \Delta] = \sum_{n+m}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + |\text{a}|[\Gamma, x, y] \geq \sum_{m}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + \sum_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + |\text{a}|[\Gamma, x, y] = |\text{let } \Delta, \Delta = \text{split}_m\ \Delta; \text{fuse}\{\bar{z} : A^{\perp} \mapsto \text{b}; x : A^{m} \mapsto \text{fuse}\{\bar{z} : A^{\perp} \mapsto \text{b}; y : A^{n} \mapsto \text{a}\}\}|[\Gamma, \Delta]$$

Commuting conversions

$\kappa\otimes$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A \mapsto \text{let } u, v = w; \text{b}\}|[\Gamma, w, \Delta] = |\text{a}|[\Gamma, x] + 3 + |\text{b}|[\Delta, y, u, v] \geq 3 + |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, u, v, y] = |\text{let } u, v = w; \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}|[\Gamma, w, \Delta]$$

$\kappa⅋$

$$|\text{fuse}\{x : C^{\perp} \mapsto \text{a}; y : C \mapsto \text{connect } w \text{ to}\{u \mapsto \text{b}; v \mapsto \text{c}\}\}|[\Gamma, w, \Delta, \xi] = |\text{a}|[\xi, x] + 3 + \chi_{+}(A) + |\text{b}|[\Gamma, u] + \chi_{+}(B) + |\text{c}|[\Delta, y, v] \geq 3 + \chi_{+}(A) + |\text{b}|[\Gamma, u] + \chi_{+}(B) + |\text{a}|[\xi, x] + |\text{c}|[\Delta, v, y] = |\text{connect } w \text{ to}\{u \mapsto \text{b}; v \mapsto \text{fuse}\{y : C \mapsto \text{c}; x : C^{\perp} \mapsto \text{a}\}\}|[\Gamma, w, \Delta, \xi]$$

$\kappa\oplus$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A \mapsto \text{case } w \text{ of}\{\text{inl } u \mapsto \text{b}; \text{inr } v \mapsto \text{c}\}\}|[\Gamma, w, \Delta] = |\text{a}|[\Gamma, x] + 3 + \text{match } w\ \{\text{inl } u \mapsto |\text{b}|[\Delta, y, u]\ ; \text{inr } v \mapsto |\text{c}|[\Delta, y, v]\} \geq 3 + \text{match } w\ \{\text{inl } u \mapsto |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, u, y]\ ; \text{inr } v \mapsto |\text{a}|[\Gamma, x] + |\text{c}|[\Delta, v, y]\} = |\text{case } w \text{ of}\{\text{inl } u \mapsto \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}; \text{inr } v \mapsto \text{fuse}\{y : A \mapsto \text{c}; x : A^{\perp} \mapsto \text{a}\}\}|[\Gamma, w, \Delta]$$

κcompare

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A \mapsto \text{compare } n, m\{\geq\ \mapsto \text{b}; \leq\ \mapsto \text{c}\}\}|[\Gamma, \Delta] = |\text{a}|[\Gamma, x] + \text{if}\ \{n \geq m \mapsto |\text{b}|[\Delta, y]; n \leq m \mapsto |\text{c}|[\Delta, y]\} \geq \text{if}\ \{n \geq m \mapsto |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, y]; n \leq m \mapsto |\text{a}|[\Gamma, x] + |\text{c}|[\Delta, y]\} = |\text{compare } n, m\{\geq\ \mapsto \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}; \leq\ \mapsto \text{fuse}\{y : A \mapsto \text{c}; x : A^{\perp} \mapsto \text{a}\}\}|[\Gamma, \Delta]$$

$\kappa\text{n}\otimes$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A^{n} \mapsto \text{let } z = \text{slice } w; \text{b}\}|[\Gamma, w, \Delta] = \sum_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + 2 + |\text{b}|[\Delta, y, z] \geq 2 + \sum_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + |\text{b}|[\Delta, z, y] = |\text{let } z = \text{slice } w; \text{fuse}\{y : A^{n} \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}|[\Gamma, w, \Delta]$$

$\kappa\text{n}⅋$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A^{n} \mapsto \text{coslice } w\{u \mapsto_{n} \text{b}; v \mapsto_{m} \text{c}\}\}|[\Gamma, \xi, w, \Delta] = \sum_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + 2 + \sum_{n}(1 + 3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(A) + \chi_{+}(B) + |\text{b}|[\Delta, y, u]) + \sum_{m}(1 + 3 \cdot \#\text{v}(\Xi) + \chi_{+}(B) + |\text{c}|[\xi, v]) \geq 2 + \sum_{n}(1 + 3 \cdot \#\text{v}(\Gamma) + 3 \cdot \#\text{v}(\Delta) + \chi_{+}(B) + |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, u, y]) + \sum_{m}(1 + 3 \cdot \#\text{v}(\Xi) + \chi_{+}(B) + |\text{c}|[\xi, v]) = |\text{coslice } w\{u \mapsto_{n} \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}; v \mapsto_{m} \text{c}\}|[\Gamma, \xi, w, \Delta]$$

$\boxed{\kappa\text{n§}}$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A^{n} \mapsto \text{traverse}\{w : \underset{n+m}{\S^{+}} B \text{ as } u \mapsto_{n} \text{b}; w : \underset{n+m}{\S^{+}} B \text{ as } v \mapsto_{m} \text{c}\}\}|[\Gamma, \xi, w, \Delta] =$$

$$\sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + n + m + 2 + 1 + \sum{}_{n}(1 + 3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(A) + |\text{b}|[\Delta, y, u]) + 1 + \sum{}_{m}(1 + 3 \cdot \#\text{v}(\Xi) + |\text{c}|[\xi, v]) \geq$$

$$n + m + 2 + 1 + \sum{}_{n}(1 + 3 \cdot \#\text{v}(\Gamma) + 3 \cdot \#\text{v}(\Delta) + |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, u, y]) + 1 + \sum{}_{m}(1 + 3 \cdot \#\text{v}(\Xi) + |\text{c}|[\xi, v]) = |\text{traverse}\{w :$$

$$\underset{n+m}{\S^{+}} B \text{ as } u \mapsto_{n} \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}; w : \underset{n+m}{\S^{+}} B \text{ as } v \mapsto_{m} \text{c}\}|[\Gamma, \xi, w, \Delta]$$

$\boxed{\kappa\text{sync}}$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A^{n} \mapsto \text{sync}\{v : B^{\perp m} \mapsto \text{b}; w : B^{mk} \mapsto \text{c}\}\}|[\Gamma, \Delta, \xi] = \sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + |\text{c}|[\xi, y, w] +$$

$$|\text{b}|[\Delta, v] + 3mk + 6m \geq \sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + |\text{c}|[\xi, w, y] + |\text{b}|[\Delta, v] + 3mk + 6m = |\text{sync}\{v : B^{\perp m} \mapsto \text{b}; w : B^{mk} \mapsto$$

$$\text{fuse}\{y : A^{n} \mapsto \text{c}; x : A^{\perp} \mapsto \text{a}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\kappa\text{loop}}$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A^{n} \mapsto \text{loop}\{w : A^{\perp} \mapsto \text{b}; u : \underset{m}{\S}(A \otimes (A^{\perp} \& 1)) \mapsto \text{c}\}\}|[\Gamma, \Delta, \xi] = \sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) +$$

$$|\text{c}|[\xi, y, u] + |\text{b}|[\Delta, w] + 7m + 6 \geq \sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Gamma) + |\text{a}|[\Gamma, x]) + |\text{c}|[\xi, u, y] + |\text{b}|[\Delta, w] + 7m + 6 = |\text{loop}\{w : A^{\perp} \mapsto \text{b}; u :$$

$$\underset{m}{\S}(A \otimes (A^{\perp} \& 1)) \mapsto \text{fuse}\{y : A^{n} \mapsto \text{c}; x : A^{\perp} \mapsto \text{a}\}\}|[\Gamma, \Delta, \xi]$$

$\boxed{\kappa\text{split}}$

$$|\text{fuse}\{z : A^{n+m} \mapsto \text{let } x, y = \text{split}_{m}\ z; \text{a}; \bar{z} : A^{\perp} \mapsto \text{b}\}|[\Gamma, \Delta] = \sum{}_{n+m}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + |\text{a}|[\Gamma, x, y] \geq$$

$$\sum{}_{m}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + \sum{}_{n}(\chi_{+}(A^{\perp}) + 3 \cdot \#\text{v}(\Delta) + |\text{b}|[\Delta, \bar{z}]) + |\text{a}|[\Gamma, x, y] = |\text{let } \Delta, \Delta = \text{split}_{m}\ \Delta; \text{fuse}\{\bar{z} :$$

$$A^{\perp} \mapsto \text{b}; x : A^{m} \mapsto \text{fuse}\{\bar{z} : A^{\perp} \mapsto \text{b}; y : A^{n} \mapsto \text{a}\}\}|[\Gamma, \Delta]$$

$\boxed{\kappa\&1}$

$$|\text{fuse}\{x : A^{\perp} \mapsto \text{a}; y : A \mapsto \text{let inl } u = w; \text{b}\}|[\Gamma, w, \Delta] = |\text{a}|[\Gamma, x] + 2 + \chi_{+}(B) + |\text{b}|[\Delta, y, u] \geq 2 + \chi_{+}(B) + |\text{a}|[\Gamma, x] + |\text{b}|[\Delta, u, y] =$$

$$|\text{let inl } u = w; \text{fuse}\{y : A \mapsto \text{b}; x : A^{\perp} \mapsto \text{a}\}|[\Gamma, w, \Delta]$$

Merge reduction: base cases

$\boxed{\text{merge-n§}}$

$$|\text{let } z = \text{merge } x, y; \text{traverse}\{w : \underset{m+n}{\S^{+}} B \text{ as } v \mapsto_{m+n} \text{a}\}|[w, x, y, \Delta] = m + n + 2 + 1 + \sum{}_{m+n}(1 + 3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(A) + |\text{a}|[\Delta, z, v]) \geq$$

$$m + n + 2 + 1 + \sum{}_{m}(1 + 3 \cdot \#\text{v}(A) + 3 \cdot \#\text{v}(\Delta) + |\text{a}|[x, \Delta, v]) + 1 + \sum{}_{n}(1 + 3 \cdot \#\text{v}(A) + 3 \cdot \#\text{v}(\Delta) + |\text{a}|[y, \Delta, v]) = |\text{let } \Delta, \Delta =$$

$$\text{split}_{m}\ \Delta; \text{traverse}\{w : \underset{m+n}{\S^{+}} B \text{ as } v \mapsto_{m} \text{a}; w : \underset{m+n}{\S^{+}} B \text{ as } v \mapsto_{n} \text{a}\}|[w, x, y, \Delta]$$

$\boxed{\text{merge-n⅋}}$

$$|\text{let } z = \text{merge } x, y; \text{coslice } w\{v \mapsto_{m+n} \text{a}\}|[w, x, y, \Delta] = m + n + 2 + \sum{}_{m+n}(3 \cdot \#\text{v}(\Delta) + 3 \cdot \#\text{v}(A) + \chi_{+}(B) + |\text{a}|[\Delta, z, v]) \geq$$

$$2 + \sum{}_{m}(1 + 3 \cdot \#\text{v}(A) + 3 \cdot \#\text{v}(\Delta) + \chi_{+}(B) + |\text{a}|[x, \Delta, v]) + \sum{}_{n}(1 + 3 \cdot \#\text{v}(A) + 3 \cdot \#\text{v}(\Delta) + \chi_{+}(B) + |\text{a}|[y, \Delta, v]) =$$

$$|\text{let } \Delta, \Delta = \text{split}_{m}\ \Delta; \text{coslice } w\{v \mapsto_{m} \text{a}; v \mapsto_{n} \text{a}\}|[w, x, y, \Delta]$$

4.3 Proof that fusion preserves semantics up to $\beta\eta$-equivalence: cases

Main reduction rules.

$$\boxed{\parr\otimes\mathrm{R}}$$

$$\left(\frac{\dfrac{\Gamma,\bar x : A^\perp \vdash \mathrm{a} \qquad \Delta,\bar y : B^\perp \vdash \mathrm{b}}{\Gamma,\Delta,\bar z : A^\perp \parr B^\perp \vdash \mathrm{connect}\ \bar z\ \mathrm{to}\{\bar x \mapsto \mathrm{a}; \bar y \mapsto \mathrm{b}\}}\parr \qquad \dfrac{\Xi, x:A, y:B \vdash \mathrm{c}}{\Xi, z : A\otimes B \vdash \mathrm{let}\ x,y = z;\mathrm{c}}\otimes}{\Gamma,\Delta,\Xi \vdash \mathrm{fuse}\{\bar z : A^\perp \parr B^\perp \mapsto \mathrm{connect}\ \bar z\ \mathrm{to}\{\bar x\mapsto \mathrm{a};\bar y\mapsto\mathrm{b}\}; z : A\otimes B \mapsto \mathrm{let}\ x,y=z;\mathrm{c}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\ z = (0,\ \lambda\bar z.\ \mathrm{let}\ \bar{z_0} =$$

$$(0,\ \lambda\_.\ \bar z)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0 = \bar{z_0}\ \mathrm{in}\ \mu_0\ s_0\ (\lambda\bar x.\ a^\bullet[\Gamma,\bar x],\ \lambda\bar y.\ b^\bullet[\Delta,\bar y]))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu = z\ \mathrm{in}\ \mu\ s\ (\lambda(x,y)\mapsto \mathrm{let}\ y_0 = (0,\ \lambda\_.\ y)\ \mathrm{in}\ \mathrm{let}\ x_0 =$$

$$(0,\ \lambda\_.\ x)\ \mathrm{in}\ c^\bullet[\xi, x_0, y_0]) \approx \mathrm{let}\ x = (0,\ \lambda\bar x.\ \mathrm{let}\ \bar{x_0} = (0,\ \lambda\_.\ \bar x)\ \mathrm{in}\ a^\bullet[\Gamma,\bar{x_0}])\ \mathrm{in}\ \mathrm{let}\ y = (0,\ \lambda\bar y.\ \mathrm{let}\ \bar{y_0} =$$

$$(0,\ \lambda\_.\ \bar y)\ \mathrm{in}\ b^\bullet[\Delta,\bar{y_0}])\ \mathrm{in}\ c^\bullet[\xi,x,y] \stackrel{def}{=} \left(\frac{\Gamma,\bar x:A^\perp\vdash \mathrm{a} \qquad \dfrac{\Delta,\bar y : B^\perp\vdash\mathrm{b}\qquad \Xi,x:A,y:B\vdash \mathrm{c}}{\Delta,\Xi,x:A\vdash \mathrm{fuse}\{\bar y:B^\perp\mapsto\mathrm{b}; y:B\mapsto\mathrm{c}\}}\textsc{Fuse}}{\Gamma,\Delta,\Xi\vdash\mathrm{fuse}\{\bar x:A^\perp\mapsto\mathrm{a}; x:A\mapsto\mathrm{fuse}\{\bar y:B^\perp\mapsto\mathrm{b};y:B\mapsto\mathrm{c}\}\}}\textsc{Fuse}\right)^\bullet$$

$$\boxed{\&\oplus 1}$$

$$\left(\frac{\dfrac{\Gamma, x : A^\perp \vdash \mathrm{a}}{\Gamma, z : A^\perp \& B^\perp \vdash \mathrm{let\ inl}\ x = z;\mathrm{a}}\&_1 \qquad \dfrac{\Delta,\bar x : A\vdash \mathrm{b} \qquad \Delta,\bar y : B\vdash\mathrm{c}}{\Delta,\bar z : A\oplus B\vdash \mathrm{case}\ \bar z\ \mathrm{of}\{\mathrm{inl}\ \bar x\mapsto \mathrm{b};\mathrm{inr}\ \bar y\mapsto\mathrm{c}\}}\oplus}{\Gamma,\Delta\vdash\mathrm{fuse}\{z : A^\perp\& B^\perp\mapsto \mathrm{let\ inl}\ x = z;\mathrm{a}; \bar z : A\oplus B\mapsto \mathrm{case}\ \bar z\ \mathrm{of}\{\mathrm{inl}\ \bar x\mapsto\mathrm{b};\mathrm{inr}\ \bar y\mapsto\mathrm{c}\}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\ \bar z = (0,\ \lambda z.\ \mathrm{let}\ z_0 =$$

$$(0,\ \lambda\_.\ z)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0 = z_0\ \mathrm{in}\ \mu_0\ s_0\ (\mathrm{inl}\ (\lambda x.\ a^\bullet[\Gamma,x])))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu = \bar z\ \mathrm{in}\ \mu\ s\ \lambda(\mathrm{inl}\ \bar x\mapsto\ \mathrm{let}\ \bar{x_0} = (0,\ \lambda\_.\ \bar x)\ \mathrm{in}\ b^\bullet[\bar{x_0},\Delta]\ |\mathrm{inr}\ \bar y\mapsto$$

$$\mathrm{let}\ \bar{y_0} = (0,\ \lambda\_.\ \bar y)\ \mathrm{in}\ c^\bullet[\bar{y_0},\Delta]\,) \approx \mathrm{let}\ \bar x = (0,\ \lambda x.\ \mathrm{let}\ x_0 = (0,\ \lambda\_.\ x)\ \mathrm{in}\ a^\bullet[\Gamma,x_0])\ \mathrm{in}\ b^\bullet[\bar x,\Delta] \stackrel{def}{=}$$

$$\left(\frac{\Gamma, x : A^\perp\vdash \mathrm{a}\qquad \Delta,\bar x : A\vdash\mathrm{b}}{\Gamma,\Delta\vdash \mathrm{fuse}\{x : A^\perp\mapsto\mathrm{a};\bar x : A\mapsto \mathrm{b}\}}\textsc{Fuse}\right)^\bullet$$

$$\boxed{\&\oplus 2}$$

$$\left(\frac{\dfrac{\Gamma, x : B^\perp \vdash \mathrm{a}}{\Gamma, z : A^\perp \& B^\perp \vdash \mathrm{let\ inr}\ x = z;\mathrm{a}}\&_2 \qquad \dfrac{\Delta,\bar x : A\vdash \mathrm{b} \qquad \Delta,\bar y : B\vdash\mathrm{c}}{\Delta,\bar z : A\oplus B\vdash \mathrm{case}\ \bar z\ \mathrm{of}\{\mathrm{inl}\ \bar x\mapsto \mathrm{b};\mathrm{inr}\ \bar y\mapsto\mathrm{c}\}}\oplus}{\Gamma,\Delta\vdash\mathrm{fuse}\{z : A^\perp\& B^\perp\mapsto \mathrm{let\ inr}\ x = z;\mathrm{a}; \bar z : A\oplus B\mapsto \mathrm{case}\ \bar z\ \mathrm{of}\{\mathrm{inl}\ \bar x\mapsto\mathrm{b};\mathrm{inr}\ \bar y\mapsto\mathrm{c}\}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\ \bar z = (0,\ \lambda z.\ \mathrm{let}\ z_0 =$$

$$(0,\ \lambda\_.\ z)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0 = z_0\ \mathrm{in}\ \mu_0\ s_0\ (\mathrm{inr}\ (\lambda x.\ a^\bullet[\Gamma,x])))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu = \bar z\ \mathrm{in}\ \mu\ s\ \lambda(\mathrm{inl}\ \bar x\mapsto\ \mathrm{let}\ \bar{x_0} = (0,\ \lambda\_.\ \bar x)\ \mathrm{in}\ b^\bullet[\bar{x_0},\Delta]\ |\mathrm{inr}\ \bar y\mapsto$$

$$\mathrm{let}\ \bar{y_0} = (0,\ \lambda\_.\ \bar y)\ \mathrm{in}\ c^\bullet[\bar{y_0},\Delta]\,) \approx \mathrm{let}\ \bar y = (0,\ \lambda x.\ \mathrm{let}\ x_0 = (0,\ \lambda\_.\ x)\ \mathrm{in}\ a^\bullet[\Gamma,x_0])\ \mathrm{in}\ c^\bullet[\bar y,\Delta] \stackrel{def}{=}$$

$$\left(\frac{\Gamma, x : B^\perp\vdash \mathrm{a}\qquad \Delta,\bar y : B\vdash\mathrm{c}}{\Gamma,\Delta\vdash \mathrm{fuse}\{x : B^\perp\mapsto\mathrm{a};\bar y : B\mapsto \mathrm{c}\}}\textsc{Fuse}\right)^\bullet$$

$$\boxed{\bot 1}$$

$$\left(\frac{\dfrac{}{z : \bot\vdash \mathrm{yield\ to}\ z}\bot \qquad \dfrac{\Gamma\vdash\mathrm{a}}{\Gamma,\bar z : 1\vdash \mathrm{let}\ \diamond = \bar z;\mathrm{a}}1}{\Gamma\vdash\mathrm{fuse}\{z : \bot\mapsto \mathrm{yield\ to}\ z;\bar z : 1\mapsto \mathrm{let}\ \diamond = \bar z;\mathrm{a}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\ \bar z = (0,\ \lambda z.\ \mathrm{let}\ z_0 = (0,\ \lambda\_.\ z)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0 =$$

$$z_0\ \mathrm{in}\ \mu_0\ s_0)\ \mathrm{in}\ \mathrm{let}\ s,\ \mu = \bar z\ \mathrm{in}\ \mu\ s\ a^\bullet[\Gamma] \approx a^\bullet[\Gamma] \stackrel{def}{=} (\ \Gamma\vdash\mathrm{a}\ )^\bullet$$

$$\boxed{\mathrm{n}\parr\otimes}$$

$$\left(\dfrac{\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \Delta, y : A^{\perp} \vdash \mathrm{c}}{\Gamma^{n}, \Delta^{m}, z : \parr_{n+m} A^{\perp} \vdash \mathrm{coslice}\ z\{x \mapsto_{n} \mathrm{a}; y \mapsto_{m} \mathrm{c}\}}\parr \qquad \dfrac{\Xi, \bar{x} : A^{n+m} \vdash \mathrm{b}}{\Xi, \bar{z} : \bigotimes_{n+m} A \vdash \mathrm{let}\ \bar{x} = \mathrm{slice}\ \bar{z}; \mathrm{b}}\otimes}{\Gamma^{n}, \Delta^{m}, \Xi \vdash \mathrm{fuse}\{z : \parr_{n+m} A^{\perp} \mapsto \mathrm{coslice}\ z\{x \mapsto_{n} \mathrm{a}; y \mapsto_{m} \mathrm{c}\}; \bar{z} : \bigotimes_{n+m} A \mapsto \mathrm{let}\ \bar{x} = \mathrm{slice}\ \bar{z}; \mathrm{b}\}}\textsc{Fuse}\right)^{\bullet} \stackrel{def}{=} \mathrm{let}\ \bar{z} =$$

$$(0,\ \lambda z.\, \mathrm{let}\ z_0 = (0,\ \lambda\_.\, z)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0 = z_0\ \mathrm{in}\ \mu_0\, s_0\, (\lambda i.\, \mathrm{if}\ 0 \le i \wedge i < n\ \mathrm{then}\ \lambda x.\, \mathrm{let}\ s_1,\ \mu_1 =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\, [\gamma_0, x])\, (i + s_1,\ \mu_1)\ \mathrm{else}\ \lambda y.\, \mathrm{let}\ s_2,\ \mu_2 = \Delta\ \mathrm{in}\ (\lambda \delta_0.\, c^{\bullet}\, [\delta_0, y])\, (-n + i + s_2,\ \mu_2)))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu = \bar{z}\ \mathrm{in}\ \mu\, s\, (\lambda \bar{x}.\, \mathrm{let}\ \bar{x_0} =$$

$$(n + m,\ \bar{x})\ \mathrm{in}\ b^{\bullet}\, [\xi, \bar{x_0}]) \approx \mathrm{let}\ x = (0,\ \lambda i_0.\, \lambda x_0.\, \mathrm{let}\ s_0,\ \mu_0 = \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\, [\gamma_0, x_0])\, (i_0 + s_0,\ \mu_0))\ \mathrm{in}\ \mathrm{let}\ y = (0,\ \lambda i.\, \lambda y_0.\, \mathrm{let}\ s,\ \mu =$$

$$\Delta\ \mathrm{in}\ (\lambda \delta_0.\, c^{\bullet}\, [\delta_0, y_0])\, (i + s,\ \mu))\ \mathrm{in}\ \mathrm{let}\ \bar{x} = (\mathrm{merge}\, m\, y\, x)\ \mathrm{in}\ b^{\bullet}\, [\xi, \bar{x}] \stackrel{def}{=}$$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A^{\perp} \vdash \mathrm{c} \qquad \dfrac{\Xi, \bar{x} : A^{n+m} \vdash \mathrm{b}}{\Xi, x : A^{n}, y : A^{m} \vdash \mathrm{let}\ \bar{x} = \mathrm{merge}\ x, y; \mathrm{b}}\textsc{Merge}_{n}}{\Delta^{m}, \Xi, x : A^{n} \vdash \mathrm{fuse}\{y : A^{\perp} \mapsto \mathrm{c}; y : A^{m} \mapsto \mathrm{let}\ \bar{x} = \mathrm{merge}\ x, y; \mathrm{b}\}}\textsc{Fuse}_{m}}{\Gamma^{n}, \Delta^{m}, \Xi \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; x : A^{n} \mapsto \mathrm{fuse}\{y : A^{\perp} \mapsto \mathrm{c}; y : A^{m} \mapsto \mathrm{let}\ \bar{x} = \mathrm{merge}\ x, y; \mathrm{b}\}\}}\textsc{Fuse}_{n}\right)^{\bullet}$$

$$\boxed{\mathrm{n}\S}$$

$$\left(\dfrac{\dfrac{\Gamma, x_1 : A^{\perp} \vdash \mathrm{a}}{\Gamma^{n+m}, x : \S^{-}_{n+m} A^{\perp} \vdash \mathrm{traverse}\{x\ \mathrm{as}\ x_1 \mapsto_{n+m} \mathrm{a}\}}\S \qquad \dfrac{\Delta, y_1 : A \vdash \mathrm{b} \qquad \Xi, y_2 : A \vdash \mathrm{c}}{\Delta^{n}, \Xi^{m}, y : \S^{+}_{n+m} A \vdash \mathrm{traverse}\{y\ \mathrm{as}\ y_1 \mapsto_{n} \mathrm{b}; y\ \mathrm{as}\ y_2 \mapsto_{m} \mathrm{c}\}}\S}{\Gamma^{n+m}, \Delta^{n}, \Xi^{m} \vdash \mathrm{fuse}\{x : \S^{-}_{n+m} A^{\perp} \mapsto \mathrm{traverse}\{x\ \mathrm{as}\ x_1 \mapsto_{n+m} \mathrm{a}\}; y : \S^{+}_{n+m} A \mapsto \mathrm{traverse}\{y\ \mathrm{as}\ y_1 \mapsto_{n} \mathrm{b}; y\ \mathrm{as}\ y_2 \mapsto_{m} \mathrm{c}\}\}}\textsc{Fuse}\right)^{\bullet} \stackrel{def}{=}$$

$$\mathrm{let}\ y = (0,\ \lambda x.\, \mathrm{let}\ x_0 = (0,\ \lambda\_.\, x)\ \mathrm{in}\ \mathrm{let}\ s_2,\ \mu_2 = x_0\ \mathrm{in}\ \mu_2\, s_2\, (\lambda i.\, \lambda x_1.\, \mathrm{let}\ s_3,\ \mu_3 = \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\, [\gamma_0, x_1])\, (i + s_3,\ \mu_3)))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu =$$

$$y\ \mathrm{in}\ \mu\, s\, (\lambda y_0.\, (\mathrm{schedule}\, (\lambda j.\, \mathrm{let}\ s_0,\ \mu_0 = \Delta\ \mathrm{in}\ (\lambda \delta_0.\, \mathrm{let}\ y_1 = (0,\ y_0\, j)\ \mathrm{in}\ b^{\bullet}\, [y_1, \delta_0])\, (j + s_0,\ \mu_0))) \gg (\mathrm{schedule}\, (\lambda j_0.\, \mathrm{let}\ s_1,\ \mu_1 =$$

$$\xi\ \mathrm{in}\ (\lambda \xi_0.\, \mathrm{let}\ y_2 = (0,\ y_0\, (n + j_0))\ \mathrm{in}\ c^{\bullet}\, [y_2, \xi_0])\, (j_0 + s_1,\ \mu_1)))) \approx \mathrm{let}\ n_0,\ \gamma_0 = \Gamma\ \mathrm{in}\ \mathrm{let}\ \gamma_1 = (n,\ \gamma_0)\ \mathrm{in}\ \mathrm{let}\ \gamma_2 =$$

$$(m,\ \gamma_0)\ \mathrm{in}\ (\mathrm{schedule}\, (\lambda j.\, \mathrm{let}\ s,\ \mu = \Delta\ \mathrm{in}\ (\lambda \delta_0.\, \mathrm{let}\ s_0,\ \mu_0 = \gamma_1\ \mathrm{in}\ (\lambda \gamma_3.\, \mathrm{let}\ y_1 = (0,\ \lambda x_1.\, \mathrm{let}\ x_{10} =$$

$$(0,\ \lambda\_.\, x_1)\ \mathrm{in}\ a^{\bullet}\, [\gamma_3, x_{10}])\ \mathrm{in}\ b^{\bullet}\, [y_1, \delta_0])\, (j + s_0,\ \mu_0))\, (j + s,\ \mu))) \gg (\mathrm{schedule}\, (\lambda j_0.\, \mathrm{let}\ s_1,\ \mu_1 = \xi\ \mathrm{in}\ (\lambda \xi_0.\, \mathrm{let}\ s_2,\ \mu_2 =$$

$$\gamma_2\ \mathrm{in}\ (\lambda \gamma_4.\, \mathrm{let}\ y_2 = (0,\ \lambda x_{11}.\, \mathrm{let}\ x_{12} = (0,\ \lambda\_.\, x_{11})\ \mathrm{in}\ a^{\bullet}\, [\gamma_4, x_{12}])\ \mathrm{in}\ c^{\bullet}\, [y_2, \xi_0])\, (j_0 + s_2,\ \mu_2))\, (j_0 + s_1,\ \mu_1))) \stackrel{def}{=}$$

$$\left(\dfrac{\dfrac{\dfrac{\Gamma, x_1 : A^{\perp} \vdash \mathrm{a} \qquad \Delta, y_1 : A \vdash \mathrm{b}}{\Delta, \Gamma \vdash \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_1 : A \mapsto \mathrm{b}\}}\textsc{Fuse} \qquad \dfrac{\Gamma, x_1 : A^{\perp} \vdash \mathrm{a} \qquad \Xi, y_2 : A \vdash \mathrm{c}}{\Xi, \Gamma \vdash \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_2 : A \mapsto \mathrm{c}\}}\textsc{Fuse}}{\Delta^{n}, \Xi^{m}, \Gamma^{n}, \Gamma^{m} \vdash \mathrm{traverse}\{\mapsto_{n} \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_1 : A \mapsto \mathrm{b}\}; \mapsto_{m} \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_2 : A \mapsto \mathrm{c}\}\}}\S}{\Gamma^{n+m}, \Delta^{n}, \Xi^{m} \vdash \mathrm{let}\ \Gamma, \Gamma = \mathrm{split}_{n}\ \Gamma; \mathrm{traverse}\{\mapsto_{n} \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_1 : A \mapsto \mathrm{b}\}; \mapsto_{m} \mathrm{fuse}\{x_1 : A^{\perp} \mapsto \mathrm{a}; y_2 : A \mapsto \mathrm{c}\}\}}\textsc{Split}_{n}\right)^{\bullet}$$

Structural reduction rules.

Associativity-L

$$\left(\dfrac{\dfrac{\Gamma, x : A^{\perp^m} \vdash \mathrm{a} \qquad \Delta, y : B^{\perp^n}, \bar{x} : A \vdash \mathrm{b}}{\Gamma, \Delta^m, y : B^{\perp^{nm}} \vdash \mathrm{cut}\{x : A^{\perp^m} \mapsto \mathrm{a}; \bar{x} : A \mapsto \mathrm{b}\}}\,\textsc{Cut}_m \qquad \Xi, \bar{y} : B \vdash \mathrm{c}}{\Gamma, \Delta^m, \Xi^{nm} \vdash \mathrm{fuse}\{y : B^{\perp^{nm}} \mapsto \mathrm{cut}\{x : A^{\perp^m} \mapsto \mathrm{a}; \bar{x} : A \mapsto \mathrm{b}\}; \bar{y} : B \mapsto \mathrm{c}\}}\,\textsc{Fuse}_{nm}\right)^\bullet \overset{def}{=} \mathrm{let}\, y = (0, \lambda i_0.\, \lambda \bar{y}.\, \mathrm{let}\, \bar{y_0} =$$

$$(0, \lambda\_.\, \lambda\kappa_0.\, \kappa_0 \propto \bar{y})\, \mathrm{in}\, \mathrm{let}\, s_1, \mu_1 = \xi\, \mathrm{in}\, (\lambda\xi_0.\, c^\bullet\, [\xi_0, \bar{y_0}])\, (i_0 + s_1, \mu_1))\, \mathrm{in}\, \mathrm{let}\, x = (0, \lambda i.\, \lambda\bar{x}.\, \mathrm{let}\, \bar{x_0} = (0, \lambda\_.\, \lambda\kappa.\, \kappa \propto$$

$$\bar{x})\, \mathrm{in}\, \mathrm{let}\, s, \mu = \Delta\, \mathrm{in}\, (\lambda\delta_0.\, \mathrm{let}\, s_0, \mu_0 = y\, \mathrm{in}\, (\lambda y_0.\, b^\bullet\, [\delta_0, \bar{x_0}, y_0])\, (ni + s_0, \mu_0))\, (i + s, \mu))\, \mathrm{in}\, a^\bullet\, [\Gamma, x] \approx \mathrm{let}\, x = (0, \lambda i.\, \lambda\bar{x}.\, \mathrm{let}\, \bar{x_0} =$$

$$(0, \lambda\_.\, \lambda\kappa_0.\, \kappa_0 \propto \bar{x})\, \mathrm{in}\, \mathrm{let}\, s, \mu = \Delta\, \mathrm{in}\, (\lambda\delta_0.\, \mathrm{let}\, s_0, \mu_0 = \xi\, \mathrm{in}\, (\lambda\xi_0.\, \mathrm{let}\, y = (0, \lambda i_0.\, \lambda\bar{y}.\, \mathrm{let}\, \bar{y_0} = (0, \lambda\_.\, \lambda\kappa.\, \kappa \propto \bar{y})\, \mathrm{in}\, \mathrm{let}\, s_1, \mu_1 =$$

$$\xi_0\, \mathrm{in}\, (\lambda\xi_1.\, c^\bullet\, [\xi_1, \bar{y_0}])\, (i_0 + s_1, \mu_1))\, \mathrm{in}\, b^\bullet\, [\delta_0, \bar{x_0}, y])\, (ni + s_0, \mu_0))\, (i + s, \mu))\, \mathrm{in}\, a^\bullet\, [\Gamma, x] \overset{def}{=}$$

$$\left(\dfrac{\Gamma, x : A^{\perp^m} \vdash \mathrm{a} \qquad \dfrac{\Delta, \bar{x} : A, y : B^{\perp^n} \vdash \mathrm{b} \qquad \Xi, \bar{y} : B \vdash \mathrm{c}}{\Delta, \Xi^n, \bar{x} : A \vdash \mathrm{fuse}\{y : B^{\perp^n} \mapsto \mathrm{b}; \bar{y} : B \mapsto \mathrm{c}\}}\,\textsc{Fuse}_n}{\Gamma, \Delta^m, \Xi^{nm} \vdash \mathrm{cut}\{x : A^{\perp^m} \mapsto \mathrm{a}; \bar{x} : A \mapsto \mathrm{fuse}\{y : B^{\perp^n} \mapsto \mathrm{b}; \bar{y} : B \mapsto \mathrm{c}\}\}}\,\textsc{Cut}_m\right)^\bullet$$

Associativity-R

$$\left(\dfrac{\dfrac{\Delta, x : A^\perp \vdash \mathrm{a} \qquad \Gamma, y : B^\perp, \bar{x} : A^n \vdash \mathrm{b}}{\Gamma, \Delta^n, y : B^\perp \vdash \mathrm{cut}\{x : A^\perp \mapsto \mathrm{a}; \bar{x} : A^n \mapsto \mathrm{b}\}}\,\textsc{Cut}_n \qquad \Xi, \bar{y} : B^m \vdash \mathrm{c}}{\Gamma^m, \Delta^{nm}, \Xi \vdash \mathrm{fuse}\{y : B^\perp \mapsto \mathrm{cut}\{x : A^\perp \mapsto \mathrm{a}; \bar{x} : A^n \mapsto \mathrm{b}\}; \bar{y} : B^m \mapsto \mathrm{c}\}}\,\textsc{Fuse}_m\right)^\bullet \overset{def}{=} \mathrm{let}\, \bar{y} = (0, \lambda i.\, \lambda y.\, \mathrm{let}\, s, \mu =$$

$$\Gamma\, \mathrm{in}\, (\lambda\gamma_0.\, \mathrm{let}\, s_0, \mu_0 = \Delta\, \mathrm{in}\, (\lambda\delta_0.\, \mathrm{let}\, \bar{x} = (0, \lambda i_0.\, \lambda x.\, \mathrm{let}\, s_1, \mu_1 = \delta_0\, \mathrm{in}\, (\lambda\delta_1.\, a^\bullet\, [\delta_1, x])\, (i_0 + s_1, \mu_1))\, \mathrm{in}\, b^\bullet\, [\gamma_0, \bar{x}, y])\, (ni +$$

$$s_0, \mu_0))\, (i + s, \mu))\, \mathrm{in}\, c^\bullet\, [\xi, \bar{y}] \approx \mathrm{let}\, \bar{x} = (0, \lambda i_0.\, \lambda x.\, \mathrm{let}\, s_1, \mu_1 = \Delta\, \mathrm{in}\, (\lambda\delta_0.\, a^\bullet\, [\delta_0, x])\, (i_0 + s_1, \mu_1))\, \mathrm{in}\, \mathrm{let}\, \bar{y} = (0, \lambda i.\, \lambda y.\, \mathrm{let}\, s, \mu =$$

$$\Gamma\, \mathrm{in}\, (\lambda\gamma_0.\, \mathrm{let}\, s_0, \mu_0 = \bar{x}\, \mathrm{in}\, (\lambda\bar{x_0}.\, b^\bullet\, [\gamma_0, \bar{x_0}, y])\, (ni + s_0, \mu_0))\, (i + s, \mu))\, \mathrm{in}\, c^\bullet\, [\xi, \bar{y}] \overset{def}{=}$$

$$\left(\dfrac{\Delta, x : A^\perp \vdash \mathrm{a} \qquad \dfrac{\Gamma, \bar{x} : A^n, y : B^\perp \vdash \mathrm{b} \qquad \Xi, \bar{y} : B^m \vdash \mathrm{c}}{\Gamma^m, \Xi, \bar{x} : A^{nm} \vdash \mathrm{fuse}\{y : B^\perp \mapsto \mathrm{b}; \bar{y} : B^m \mapsto \mathrm{c}\}}\,\textsc{Fuse}_m}{\Gamma^m, \Delta^{nm}, \Xi \vdash \mathrm{cut}\{x : A^\perp \mapsto \mathrm{a}; \bar{x} : A^{nm} \mapsto \mathrm{fuse}\{y : B^\perp \mapsto \mathrm{b}; \bar{y} : B^m \mapsto \mathrm{c}\}\}}\,\textsc{Cut}_{nm}\right)^\bullet$$

Ax

$$\left(\dfrac{\Gamma, x : A^\perp \vdash \mathrm{a}[x] \qquad \dfrac{}{w : A^\perp, y : A \vdash y \leftrightarrow w}\,\textsc{Ax}}{\Gamma, w : A^\perp \vdash \mathrm{fuse}\{x : A^\perp \mapsto \mathrm{a}[x]; y : A \mapsto y \leftrightarrow w\}}\,\textsc{Fuse}\right)^\bullet \overset{def}{=} \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 = (0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [x_0, \Gamma])\, \mathrm{in}\, \mathrm{let}\, s, \mu =$$

$$y\, \mathrm{in}\, \mathrm{let}\, s_0, \mu_0 = w\, \mathrm{in}\, \mu\, s\, (\mu_0\, s_0) \approx a^\bullet\, [w, \Gamma] \overset{def}{=} \left(\, \Gamma, w : A^\perp \vdash \mathrm{a}[w] \,\right)^\bullet$$

Split

$$\left(\dfrac{\Delta, \bar{z} : A^\perp \vdash \mathrm{b} \qquad \dfrac{\Gamma, x : A^m, y : A^n \vdash \mathrm{a}}{\Gamma, z : A^{n+m} \vdash \mathrm{let}\, x, y = \mathrm{split}_m\, z; \mathrm{a}}\,\textsc{Split}_m}{\Gamma, \Delta^{n+m} \vdash \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; z : A^{n+m} \mapsto \mathrm{let}\, x, y = \mathrm{split}_m\, z; \mathrm{a}\}}\,\textsc{Fuse}_{n+m}\right)^\bullet \overset{def}{=} \mathrm{let}\, z = (0, \lambda i.\, \lambda\bar{z}.\, \mathrm{let}\, s, \mu =$$

$$\Delta\, \mathrm{in}\, (\lambda\delta_0.\, b^\bullet\, [\delta_0, \bar{z}])\, (i + s, \mu))\, \mathrm{in}\, \mathrm{let}\, n_0, z_0 = z\, \mathrm{in}\, \mathrm{let}\, x = (m, z_0)\, \mathrm{in}\, \mathrm{let}\, y = (n, z_0)\, \mathrm{in}\, a^\bullet\, [x, y, \Gamma] \approx \mathrm{let}\, n_0, \delta_0 = \Delta\, \mathrm{in}\, \mathrm{let}\, \delta_1 =$$

$$(m, \delta_0)\, \mathrm{in}\, \mathrm{let}\, \delta_2 = (n, \delta_0)\, \mathrm{in}\, \mathrm{let}\, x = (0, \lambda i_0.\, \lambda\bar{z_0}.\, \mathrm{let}\, s_0, \mu_0 = \delta_1\, \mathrm{in}\, (\lambda\delta_4.\, b^\bullet\, [\delta_4, \bar{z_0}])\, (i_0 + s_0, \mu_0))\, \mathrm{in}\, \mathrm{let}\, y = (0, \lambda i.\, \lambda\bar{z}.\, \mathrm{let}\, s, \mu =$$

$$\delta_2\, \mathrm{in}\, (\lambda\delta_3.\, b^\bullet\, [\delta_3, \bar{z}])\, (i + s, \mu))\, \mathrm{in}\, a^\bullet\, [x, y, \Gamma] \overset{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, \bar{z} : A^\perp \vdash \mathrm{b} \qquad \dfrac{\Delta, \bar{z} : A^\perp \vdash \mathrm{b} \qquad \Gamma, x : A^m, y : A^n \vdash \mathrm{a}}{\Gamma, \Delta^n, x : A^m \vdash \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}}\,\textsc{Fuse}_n}{\Gamma, \Delta^m, \Delta^n \vdash \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; x : A^m \mapsto \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}\}}\,\textsc{Fuse}_m}{\Gamma, \Delta^{n+m} \vdash \mathrm{let}\, \Delta, \Delta = \mathrm{split}_m\, \Delta; \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; x : A^m \mapsto \mathrm{fuse}\{\bar{z} : A^\perp \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}\}}\,\textsc{Split}_m\right)^\bullet$$

Commuting conversions

$\boxed{\kappa\otimes}$

$$\left(\dfrac{\Gamma, x : A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A, u : B, v : C \vdash \mathrm{b}}{w : B \otimes C, \Delta, y : A \vdash \mathrm{let}\, u, v = w; \mathrm{b}}\otimes}{\Gamma, w : B \otimes C, \Delta \vdash \mathrm{fuse}\{x : A^\perp \mapsto \mathrm{a}; y : A \mapsto \mathrm{let}\, u, v = w; \mathrm{b}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 = (0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\Gamma, x_0])\, \mathrm{in}\, \mathrm{let}\, s,\ \mu =$$

$$w\, \mathrm{in}\, \mu\, s\, (\lambda(u, v) \mapsto \mathrm{let}\, v_0 = (0, \lambda\_.\, v)\, \mathrm{in}\, \mathrm{let}\, u_0 = (0, \lambda\_.\, u)\, \mathrm{in}\, b^\bullet\, [\Delta, u_0, v_0, y]) \approx \mathrm{let}\, s,\ \mu = w\, \mathrm{in}\, \mu\, s\, (\lambda(u, v) \mapsto \mathrm{let}\, v_0 =$$

$$(0, \lambda\_.\, v)\, \mathrm{in}\, \mathrm{let}\, u_0 = (0, \lambda\_.\, u)\, \mathrm{in}\, \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 = (0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\Gamma, x_0])\, \mathrm{in}\, b^\bullet\, [\Delta, u_0, v_0, y]) \stackrel{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, u : B, v : C, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, u : B, v : C \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^\perp \mapsto \mathrm{a}\}}\textsc{Fuse}}{\Gamma, w : B \otimes C, \Delta \vdash \mathrm{let}\, u, v = w; \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^\perp \mapsto \mathrm{a}\}}\otimes\right)^\bullet$$

$\boxed{\kappa⅋}$

$$\left(\dfrac{\Xi, x : C^\perp \vdash \mathrm{a} \qquad \dfrac{\Gamma, u : A \vdash \mathrm{b} \qquad \Delta, y : C, v : B \vdash \mathrm{c}}{\Gamma, w : A ⅋ B, \Delta, y : C \vdash \mathrm{connect}\, w\, \mathrm{to}\{u \mapsto \mathrm{b}; v \mapsto \mathrm{c}\}}⅋}{\Gamma, w : A ⅋ B, \Delta, \Xi \vdash \mathrm{fuse}\{x : C^\perp \mapsto \mathrm{a}; y : C \mapsto \mathrm{connect}\, w\, \mathrm{to}\{u \mapsto \mathrm{b}; v \mapsto \mathrm{c}\}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 =$$

$$(0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\xi, x_0])\, \mathrm{in}\, \mathrm{let}\, s,\ \mu = w\, \mathrm{in}\, \mu\, s\, (\lambda u.\, \mathrm{let}\, u_0 = (0, \lambda\_.\, \lambda\kappa.\, \kappa \propto u)\, \mathrm{in}\, b^\bullet\, [\Gamma, u_0],\ \lambda v.\, \mathrm{let}\, v_0 = (0, \lambda\_.\, \lambda\kappa_0.\, \kappa_0 \propto$$

$$v)\, \mathrm{in}\, c^\bullet\, [\Delta, v_0, y]) \approx \mathrm{let}\, s,\ \mu = w\, \mathrm{in}\, \mu\, s\, (\lambda u.\, \mathrm{let}\, u_0 = (0, \lambda\_.\, \lambda\kappa.\, \kappa \propto u)\, \mathrm{in}\, b^\bullet\, [\Gamma, u_0],\ \lambda v.\, \mathrm{let}\, v_0 = (0, \lambda\_.\, \lambda\kappa_0.\, \kappa_0 \propto v)\, \mathrm{in}\, \mathrm{let}\, y =$$

$$(0, \lambda x.\, \mathrm{let}\, x_0 = (0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\xi, x_0])\, \mathrm{in}\, c^\bullet\, [\Delta, v_0, y]) \stackrel{def}{=}$$

$$\left(\dfrac{\Gamma, u : A \vdash \mathrm{b} \qquad \dfrac{\Delta, v : B, y : C \vdash \mathrm{c} \qquad \Xi, x : C^\perp \vdash \mathrm{a}}{\Delta, \Xi, v : B \vdash \mathrm{fuse}\{y : C \mapsto \mathrm{c}; x : C^\perp \mapsto \mathrm{a}\}}\textsc{Fuse}}{\Gamma, w : A ⅋ B, \Delta, \Xi \vdash \mathrm{connect}\, w\, \mathrm{to}\{u \mapsto \mathrm{b}; v \mapsto \mathrm{fuse}\{y : C \mapsto \mathrm{c}; x : C^\perp \mapsto \mathrm{a}\}\}}⅋\right)^\bullet$$

$\boxed{\kappa\oplus}$

$$\left(\dfrac{\Gamma, x : A^\perp \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A, u : B \vdash \mathrm{b} \qquad \Delta, y : A, v : C \vdash \mathrm{c}}{w : B \oplus C, \Delta, y : A \vdash \mathrm{case}\, w\, \mathrm{of}\{\mathrm{inl}\, u \mapsto \mathrm{b}; \mathrm{inr}\, v \mapsto \mathrm{c}\}}\oplus}{\Gamma, w : B \oplus C, \Delta \vdash \mathrm{fuse}\{x : A^\perp \mapsto \mathrm{a}; y : A \mapsto \mathrm{case}\, w\, \mathrm{of}\{\mathrm{inl}\, u \mapsto \mathrm{b}; \mathrm{inr}\, v \mapsto \mathrm{c}\}\}}\textsc{Fuse}\right)^\bullet \stackrel{def}{=} \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 =$$

$$(0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\Gamma, x_0])\, \mathrm{in}\, \mathrm{let}\, s,\ \mu = w\, \mathrm{in}\, \mu\, s\, \lambda(\mathrm{inl}\, u \mapsto\ \mathrm{let}\, u_0 = (0, \lambda\_.\, u)\, \mathrm{in}\, b^\bullet\, [\Delta, u_0, y]\, |\mathrm{inr}\, v \mapsto\ \mathrm{let}\, v_0 =$$

$$(0, \lambda\_.\, v)\, \mathrm{in}\, c^\bullet\, [\Delta, v_0, y]\,) \approx \mathrm{let}\, s,\ \mu = w\, \mathrm{in}\, \mu\, s\, \lambda(\mathrm{inl}\, u \mapsto\ \mathrm{let}\, u_0 = (0, \lambda\_.\, u)\, \mathrm{in}\, \mathrm{let}\, y = (0, \lambda x.\, \mathrm{let}\, x_0 =$$

$$(0, \lambda\_.\, x)\, \mathrm{in}\, a^\bullet\, [\Gamma, x_0])\, \mathrm{in}\, b^\bullet\, [\Delta, u_0, y]\, |\mathrm{inr}\, v \mapsto\ \mathrm{let}\, v_0 = (0, \lambda\_.\, v)\, \mathrm{in}\, \mathrm{let}\, y_0 = (0, \lambda x_1.\, \mathrm{let}\, x_2 =$$

$$(0, \lambda\_.\, x_1)\, \mathrm{in}\, a^\bullet\, [\Gamma, x_2])\, \mathrm{in}\, c^\bullet\, [\Delta, v_0, y_0]\,) \stackrel{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, u : B, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, u : B \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^\perp \mapsto \mathrm{a}\}}\textsc{Fuse} \qquad \dfrac{\Delta, v : C, y : A \vdash \mathrm{c} \qquad \Gamma, x : A^\perp \vdash \mathrm{a}}{\Gamma, \Delta, v : C \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{c}; x : A^\perp \mapsto \mathrm{a}\}}\textsc{Fuse}}{\Gamma, w : B \oplus C, \Delta \vdash \mathrm{case}\, w\, \mathrm{of}\{\mathrm{inl}\, u \mapsto \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^\perp \mapsto \mathrm{a}\}; \mathrm{inr}\, v \mapsto \mathrm{fuse}\{y : A \mapsto \mathrm{c}; x : A^\perp \mapsto \mathrm{a}\}\}}\oplus\right)^\bullet$$

$\boxed{\kappa\text{compare}}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A \vdash \mathrm{b} \qquad \Delta, y : A \vdash \mathrm{c}}{\Delta, y : A \vdash \mathrm{compare}\ n, m\{\geq\ \mapsto \mathrm{b}; \leq\ \mapsto \mathrm{c}\}}\,\textsc{compare}}{\Gamma, \Delta \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A \mapsto \mathrm{compare}\ n, m\{\geq\ \mapsto \mathrm{b}; \leq\ \mapsto \mathrm{c}\}\}}\,\textsc{Fuse}\right)^{\bullet} \overset{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda x.\, \mathrm{let}\ x_0\ =$$

$$(0,\ \lambda\_.\, x)\ \mathrm{in}\ a^{\bullet}\,[\Gamma, x_0])\ \mathrm{in}\ \mathrm{if}\ n\ \geq\ m\ \mathrm{then}\ b^{\bullet}\,[\Delta, y]\ \mathrm{else}\ c^{\bullet}\,[\Delta, y] \approx \mathrm{if}\ n\ \geq\ m\ \mathrm{then}\ \mathrm{let}\ y\ =\ (0,\ \lambda x.\, \mathrm{let}\ x_0\ =$$

$$(0,\ \lambda\_.\, x)\ \mathrm{in}\ a^{\bullet}\,[\Gamma, x_0])\ \mathrm{in}\ b^{\bullet}\,[\Delta, y]\ \mathrm{else}\ \mathrm{let}\ y_0\ =\ (0,\ \lambda x_1.\, \mathrm{let}\ x_2\ =\ (0,\ \lambda\_.\, x_1)\ \mathrm{in}\ a^{\bullet}\,[\Gamma, x_2])\ \mathrm{in}\ c^{\bullet}\,[\Delta, y_0] \overset{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\,\textsc{Fuse} \qquad \dfrac{\Delta, y : A \vdash \mathrm{c} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}}\,\textsc{Fuse}}{\Gamma, \Delta \vdash \mathrm{compare}\ n, m\{\geq\ \mapsto \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}; \leq\ \mapsto \mathrm{fuse}\{y : A \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}\}}\,\textsc{compare}\right)^{\bullet}$$

$\boxed{\kappa\text{n}\otimes}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A^{n}, z : B^{n} \vdash \mathrm{b}}{w : \bigotimes_{n} B, \Delta, y : A^{n} \vdash \mathrm{let}\ z = \mathrm{slice}\ w; \mathrm{b}}\,\otimes}{\Gamma^{n}, w : \bigotimes_{n} B, \Delta \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A^{n} \mapsto \mathrm{let}\ z = \mathrm{slice}\ w; \mathrm{b}\}}\,\textsc{Fuse}_{n}\right)^{\bullet} \overset{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda i.\, \lambda x.\, \mathrm{let}\ s_0,\ \mu_0\ =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\,[\gamma_0, x])\ (i + s_0,\ \mu_0))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda z.\, \mathrm{let}\ z_0\ =\ (n,\ z)\ \mathrm{in}\ b^{\bullet}\,[\Delta, z_0, y]) \approx \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda z.\, \mathrm{let}\ z_0\ =$$

$$(n,\ z)\ \mathrm{in}\ \mathrm{let}\ y\ =\ (0,\ \lambda i.\, \lambda x.\, \mathrm{let}\ s_0,\ \mu_0\ =\ \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\,[\gamma_0, x])\ (i + s_0,\ \mu_0))\ \mathrm{in}\ b^{\bullet}\,[\Delta, z_0, y]) \overset{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, z : B^{n}, y : A^{n} \vdash \mathrm{b} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma^{n}, \Delta, z : B^{n} \vdash \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\,\textsc{Fuse}_{n}}{\Gamma^{n}, w : \bigotimes_{n} B, \Delta \vdash \mathrm{let}\ z = \mathrm{slice}\ w; \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\,\otimes\right)^{\bullet}$$

$\boxed{\kappa\text{n}⅋}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A, u : B \vdash \mathrm{b} \qquad \Xi, v : B \vdash \mathrm{c}}{\Xi^{m}, w : ⅋_{n+m} B, \Delta^{n}, y : A^{n} \vdash \mathrm{coslice}\ w\{u \mapsto_{n} \mathrm{b}; v \mapsto_{m} \mathrm{c}\}}\,⅋}{\Gamma^{n}, \Xi^{m}, w : ⅋_{n+m} B, \Delta^{n} \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A^{n} \mapsto \mathrm{coslice}\ w\{u \mapsto_{n} \mathrm{b}; v \mapsto_{m} \mathrm{c}\}\}}\,\textsc{Fuse}_{n}\right)^{\bullet} \overset{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda i_0.\, \lambda x.\, \mathrm{let}\ s_3,\ \mu_3\ =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, a^{\bullet}\,[\gamma_0, x])\ (i_0 + s_3,\ \mu_3))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda i.\, \mathrm{if}\ 0\ \leq\ i\ \wedge\ i\ <\ n\ \mathrm{then}\ \lambda u.\, \mathrm{let}\ u_0\ =\ (0,\ \lambda\_.\, \lambda \kappa.\, \kappa\ \propto\ u)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0\ =$$

$$\Delta\ \mathrm{in}\ (\lambda \delta_0.\, \mathrm{let}\ s_1,\ \mu_1\ =\ y\ \mathrm{in}\ (\lambda y_0.\, b^{\bullet}\,[\delta_0, u_0, y_0])\ (i + s_1,\ \mu_1))\ (i + s_0,\ \mu_0)\ \mathrm{else}\ \lambda v.\, \mathrm{let}\ v_0\ =\ (0,\ \lambda\_.\, \lambda \kappa_0.\, \kappa_0\ \propto\ v)\ \mathrm{in}\ \mathrm{let}\ s_2,\ \mu_2\ =$$

$$\xi\ \mathrm{in}\ (\lambda \xi_0.\, c^{\bullet}\,[v_0, \xi_0])\ (-n + i + s_2,\ \mu_2)) \approx \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda i.\, \mathrm{if}\ 0\ \leq\ i\ \wedge\ i\ <\ n\ \mathrm{then}\ \lambda u.\, \mathrm{let}\ u_0\ =\ (0,\ \lambda\_.\, \lambda \kappa.\, \kappa\ \propto$$

$$u)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0\ =\ \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\, \mathrm{let}\ s_1,\ \mu_1\ =\ \Delta\ \mathrm{in}\ (\lambda \delta_0.\, \mathrm{let}\ y\ =\ (0,\ \lambda x.\, \mathrm{let}\ x_0\ =$$

$$(0,\ \lambda\_.\, x)\ \mathrm{in}\ a^{\bullet}\,[\gamma_0, x_0])\ \mathrm{in}\ b^{\bullet}\,[\delta_0, u_0, y])\ (i + s_1,\ \mu_1))\ (i + s_0,\ \mu_0)\ \mathrm{else}\ \lambda v.\, \mathrm{let}\ v_0\ =\ (0,\ \lambda\_.\, \lambda \kappa_0.\, \kappa_0\ \propto\ v)\ \mathrm{in}\ \mathrm{let}\ s_2,\ \mu_2\ =$$

$$\xi\ \mathrm{in}\ (\lambda \xi_0.\, c^{\bullet}\,[v_0, \xi_0])\ (-n + i + s_2,\ \mu_2)) \overset{def}{=} \left(\dfrac{\dfrac{\Delta, u : B, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta, u : B \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\,\textsc{Fuse} \qquad \Xi, v : B \vdash \mathrm{c}}{\Gamma^{n}, \Xi^{m}, w : ⅋_{n+m} B, \Delta^{n} \vdash \mathrm{coslice}\ w\{u \mapsto_{n} \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}; v \mapsto_{m} \mathrm{c}\}}\,⅋\right)^{\bullet}$$

$\boxed{\kappa\mathrm{n}\S}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A, u : B \vdash \mathrm{b} \qquad \Xi, v : B \vdash \mathrm{c}}{\Xi^{m}, w : \S^{+}{}_{n+m} B, \Delta^{n}, y : A^{n} \vdash \mathrm{traverse}\{w \text{ as } u \mapsto_{n} \mathrm{b}; w \text{ as } v \mapsto_{m} \mathrm{c}\}}\ \S}{\Gamma^{n}, \Xi^{m}, w : \S^{+}{}_{n+m} B, \Delta^{n} \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A^{n} \mapsto \mathrm{traverse}\{w \text{ as } u \mapsto_{n} \mathrm{b}; w \text{ as } v \mapsto_{m} \mathrm{c}\}\}}\ \textsc{Fuse}_{n}\right)^{\bullet} \stackrel{def}{=} \mathrm{let}\ y\ =$$

$$(0,\ \lambda i.\ \lambda x.\ \mathrm{let}\ s_3,\ \mu_3\ =\ \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ a^{\bullet}\,[\gamma_0, x])\ (i + s_3,\ \mu_3))\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda w_0.\ (\mathrm{schedule}\ (\lambda j.\ \mathrm{let}\ s_0,\ \mu_0\ =$$

$$\Delta\ \mathrm{in}\ (\lambda \delta_0.\ \mathrm{let}\ s_1,\ \mu_1\ =\ y\ \mathrm{in}\ (\lambda y_0.\ \mathrm{let}\ u\ =\ (0,\ w_0\ j)\ \mathrm{in}\ b^{\bullet}\,[\delta_0, u, y_0])\ (j + s_1,\ \mu_1))\ (j + s_0,\ \mu_0)))\ \gg\ (\mathrm{schedule}\ (\lambda j_0.\ \mathrm{let}\ s_2,\ \mu_2\ =$$

$$\xi\ \mathrm{in}\ (\lambda \xi_0.\ \mathrm{let}\ v\ =\ (0,\ w_0\ (n + j_0))\ \mathrm{in}\ c^{\bullet}\,[v, \xi_0])\ (j_0 + s_2,\ \mu_2)))) \approx \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\lambda w_0.\ (\mathrm{schedule}\ (\lambda j.\ \mathrm{let}\ s_0,\ \mu_0\ =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ \mathrm{let}\ s_1,\ \mu_1\ =\ \Delta\ \mathrm{in}\ (\lambda \delta_0.\ \mathrm{let}\ u\ =\ (0,\ w_0\ j)\ \mathrm{in}\ \mathrm{let}\ y\ =\ (0,\ \lambda x.\ \mathrm{let}\ x_0\ =\ (0,\ \lambda\_.\ x)\ \mathrm{in}\ a^{\bullet}\,[\gamma_0, x_0])\ \mathrm{in}\ b^{\bullet}\,[\delta_0, u, y])\ (j +$$

$$s_1,\ \mu_1))\ (j + s_0,\ \mu_0)))\ \gg\ (\mathrm{schedule}\ (\lambda j_0.\ \mathrm{let}\ s_2,\ \mu_2\ =\ \xi\ \mathrm{in}\ (\lambda \xi_0.\ \mathrm{let}\ v\ =\ (0,\ w_0\ (n + j_0))\ \mathrm{in}\ c^{\bullet}\,[v, \xi_0])\ (j_0 + s_2,\ \mu_2)))) \stackrel{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, u : B, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta, u : B \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\ \textsc{Fuse} \qquad \Xi, v : B \vdash \mathrm{c}}{\Gamma^{n}, \Xi^{m}, w : \S^{+}{}_{n+m} B, \Delta^{n} \vdash \mathrm{traverse}\{w \text{ as } u \mapsto_{n} \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}; w \text{ as } v \mapsto_{m} \mathrm{c}\}}\ \S\right)^{\bullet}$$

$\boxed{\kappa\mathrm{sync}}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, v : B^{\perp m} \vdash \mathrm{b} \qquad \Xi, y : A^{n}, w : B^{mk} \vdash \mathrm{c}}{\Delta, \Xi, y : A^{n} \vdash \mathrm{sync}\{v : B^{\perp m} \mapsto \mathrm{b}; w : B^{mk} \mapsto \mathrm{c}\}}\ \textsc{sync}^{k}_{m}}{\Gamma^{n}, \Delta, \Xi \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A^{n} \mapsto \mathrm{sync}\{v : B^{\perp m} \mapsto \mathrm{b}; w : B^{mk} \mapsto \mathrm{c}\}\}}\ \textsc{Fuse}_{n}\right)^{\bullet} \stackrel{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda i_1.\ \lambda x.\ \mathrm{let}\ s,\ \mu\ =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ a^{\bullet}\,[\gamma_0, x])\ (i_1 + s,\ \mu))\ \mathrm{in}\ \mathrm{allocate}\ (\lambda d_0.\ \ldots\ (\mathrm{allocate}\ (\lambda d_{m-1}.\ (\mathrm{let}\ v\ =\ (0,\ \lambda\_.\ (m,\ \lambda i.\ \mathrm{write}\ d_i))\ \mathrm{in}\ b^{\bullet}\,[\Delta, v])\ \gg\ (\mathrm{let}\ w\ =$$

$$(m,\ \lambda i_0.\ \mathrm{read}\ d_i)\ \mathrm{in}\ c^{\bullet}\,[\xi, w, y])))) \approx \mathrm{allocate}\ (\lambda d_0.\ \ldots\ (\mathrm{allocate}\ (\lambda d_{m-1}.\ (\mathrm{let}\ v\ =\ (0,\ \lambda\_.\ (m,\ \lambda i.\ \mathrm{write}\ d_i))\ \mathrm{in}\ b^{\bullet}\,[\Delta, v])\ \gg\ (\mathrm{let}\ w\ =$$

$$(m,\ \lambda i_1.\ \mathrm{read}\ d_i)\ \mathrm{in}\ \mathrm{let}\ y\ =\ (0,\ \lambda i_0.\ \lambda x.\ \mathrm{let}\ s,\ \mu\ =\ \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ a^{\bullet}\,[\gamma_0, x])\ (i_0 + s,\ \mu))\ \mathrm{in}\ c^{\bullet}\,[\xi, w, y])))) \stackrel{def}{=}$$

$$\left(\dfrac{\Delta, v : B^{\perp m} \vdash \mathrm{b} \qquad \dfrac{\Xi, w : B^{mk}, y : A^{n} \vdash \mathrm{c} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma^{n}, \Xi, w : B^{mk} \vdash \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}}\ \textsc{Fuse}_{n}}{\Gamma^{n}, \Delta, \Xi \vdash \mathrm{sync}\{v : B^{\perp m} \mapsto \mathrm{b}; w : B^{mk} \mapsto \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}\}}\ \textsc{sync}^{k}_{m}\right)^{\bullet}$$

$\boxed{\kappa\mathrm{loop}}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, w : A^{\perp} \vdash \mathrm{b} \qquad \Xi, y : A^{n}, u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)) \vdash \mathrm{c}}{\Delta, \Xi, y : A^{n} \vdash \mathrm{loop}\{w : A^{\perp} \mapsto \mathrm{b}; u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)) \mapsto \mathrm{c}\}}\ \textsc{loop}}{\Gamma^{n}, \Delta, \Xi \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A^{n} \mapsto \mathrm{loop}\{w : A^{\perp} \mapsto \mathrm{b}; u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)) \mapsto \mathrm{c}\}\}}\ \textsc{Fuse}_{n}\right)^{\bullet} \stackrel{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda i_0.\ \lambda x.\ \mathrm{let}\ s,\ \mu\ =$$

$$\Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ a^{\bullet}\,[\gamma_0, x])\ (i_0 + s,\ \mu))\ \mathrm{in}\ \mathrm{allocate}\ (\lambda d.\ (\mathrm{let}\ w\ =\ (0,\ \lambda\_.\ \mathrm{write}\ d)\ \mathrm{in}\ b^{\bullet}\,[\Delta, w])\ \gg\ (\mathrm{let}\ u\ =$$

$$(m,\ \lambda i.\ \lambda u_0.\ \mathrm{read}\ d\ (\lambda r.\ u_0\ (r,\ \lambda u_1.\ \mathrm{case}\ u_1\ \mathrm{of}\ \mathrm{inl}\ w_0 \mapsto\ \mathrm{write}\ d\ |\mathrm{inr}\ o \mapsto\ o)))\ \mathrm{in}\ c^{\bullet}\,[\xi, u, y])) \approx \mathrm{allocate}\ (\lambda d.\ (\mathrm{let}\ w\ =$$

$$(0,\ \lambda\_.\ \mathrm{write}\ d)\ \mathrm{in}\ b^{\bullet}\,[\Delta, w])\ \gg\ (\mathrm{let}\ u\ =\ (m,\ \lambda i_0.\ \lambda u_0.\ \mathrm{read}\ d\ (\lambda r.\ u_0\ (r,\ \lambda u_1.\ \mathrm{case}\ u_1\ \mathrm{of}\ \mathrm{inl}\ w_0 \mapsto\ \mathrm{write}\ d\ |\mathrm{inr}\ o \mapsto\ o)))\ \mathrm{in}\ \mathrm{let}\ y\ =$$

$$(0,\ \lambda i.\ \lambda x.\ \mathrm{let}\ s,\ \mu\ =\ \Gamma\ \mathrm{in}\ (\lambda \gamma_0.\ a^{\bullet}\,[\gamma_0, x])\ (i + s,\ \mu))\ \mathrm{in}\ c^{\bullet}\,[\xi, u, y])) \stackrel{def}{=}$$

$$\left(\dfrac{\Delta, w : A^{\perp} \vdash \mathrm{b} \qquad \dfrac{\Xi, u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)), y : A^{n} \vdash \mathrm{c} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma^{n}, \Xi, u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)) \vdash \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}}\ \textsc{Fuse}_{n}}{\Gamma^{n}, \Delta, \Xi \vdash \mathrm{loop}\{w : A^{\perp} \mapsto \mathrm{b}; u : \S_{m}(A \otimes (A^{\perp} \,\&\, 1)) \mapsto \mathrm{fuse}\{y : A^{n} \mapsto \mathrm{c}; x : A^{\perp} \mapsto \mathrm{a}\}\}}\ \textsc{loop}\right)^{\bullet}$$

$\boxed{\kappa\text{split}}$

$$\left(\dfrac{\dfrac{\Gamma, x : A^m, y : A^n \vdash \mathrm{a}}{\Gamma, z : A^{n+m} \vdash \mathrm{let}\ x, y = \mathrm{split}_m\ z; \mathrm{a}}\ \textsc{Split}_m \qquad \Delta, \bar{z} : A^{\perp} \vdash \mathrm{b}}{\Gamma, \Delta^{n+m} \vdash \mathrm{fuse}\{z : A^{n+m} \mapsto \mathrm{let}\ x, y = \mathrm{split}_m\ z; \mathrm{a}; \bar{z} : A^{\perp} \mapsto \mathrm{b}\}}\ \textsc{Fuse}_{n+m}\right)^{\bullet} \overset{def}{=} \mathrm{let}\ z\ =\ (0,\ \lambda i.\ \lambda \bar{z}.\ \mathrm{let}\ s,\ \mu\ =$$

$$\Delta\ \mathrm{in}\ (\lambda \delta_0.\ b^{\bullet}\ [\delta_0, \bar{z}])\ (i + s,\ \mu))\ \mathrm{in}\ \mathrm{let}\ n_0,\ z_0\ =\ z\ \mathrm{in}\ \mathrm{let}\ x\ =\ (m,\ z_0)\ \mathrm{in}\ \mathrm{let}\ y\ =\ (n,\ z_0)\ \mathrm{in}\ a^{\bullet}\ [x, y, \Gamma] \approx \mathrm{let}\ n_0,\ \delta_0\ =\ \Delta\ \mathrm{in}\ \mathrm{let}\ \delta_1\ =$$

$$(m,\ \delta_0)\ \mathrm{in}\ \mathrm{let}\ \delta_2\ =\ (n,\ \delta_0)\ \mathrm{in}\ \mathrm{let}\ x\ =\ (0,\ \lambda i_0.\ \lambda \bar{z_0}.\ \mathrm{let}\ s_0,\ \mu_0\ =\ \delta_1\ \mathrm{in}\ (\lambda \delta_4.\ b^{\bullet}\ [\delta_4, \bar{z_0}])\ (i_0 + s_0,\ \mu_0))\ \mathrm{in}\ \mathrm{let}\ y\ =\ (0,\ \lambda i.\ \lambda \bar{z}.\ \mathrm{let}\ s,\ \mu\ =$$

$$\delta_2\ \mathrm{in}\ (\lambda \delta_3.\ b^{\bullet}\ [\delta_3, \bar{z}])\ (i + s,\ \mu))\ \mathrm{in}\ a^{\bullet}\ [x, y, \Gamma] \overset{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, \bar{z} : A^{\perp} \vdash \mathrm{b} \qquad \dfrac{\Delta, \bar{z} : A^{\perp} \vdash \mathrm{b} \qquad \Gamma, x : A^m, y : A^n \vdash \mathrm{a}}{\Gamma, \Delta^n, x : A^m \vdash \mathrm{fuse}\{\bar{z} : A^{\perp} \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}}\ \textsc{Fuse}_n}{\Gamma, \Delta^m, \Delta^n \vdash \mathrm{fuse}\{\bar{z} : A^{\perp} \mapsto \mathrm{b}; x : A^m \mapsto \mathrm{fuse}\{\bar{z} : A^{\perp} \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}\}}\ \textsc{Fuse}_m}{\Gamma, \Delta^{n+m} \vdash \mathrm{let}\ \Delta, \Delta = \mathrm{split}_m\ \Delta; \mathrm{fuse}\{\bar{z} : A^{\perp} \mapsto \mathrm{b}; x : A^m \mapsto \mathrm{fuse}\{\bar{z} : A^{\perp} \mapsto \mathrm{b}; y : A^n \mapsto \mathrm{a}\}\}}\ \textsc{Split}_m\right)^{\bullet}$$

$\boxed{\kappa\&1}$

$$\left(\dfrac{\Gamma, x : A^{\perp} \vdash \mathrm{a} \qquad \dfrac{\Delta, y : A, u : B \vdash \mathrm{b}}{w : B \,\&\, C, \Delta, y : A \vdash \mathrm{let\ inl}\ u = w; \mathrm{b}}\ \&_1}{\Gamma, w : B \,\&\, C, \Delta \vdash \mathrm{fuse}\{x : A^{\perp} \mapsto \mathrm{a}; y : A \mapsto \mathrm{let\ inl}\ u = w; \mathrm{b}\}}\ \textsc{Fuse}\right)^{\bullet} \overset{def}{=} \mathrm{let}\ y\ =\ (0,\ \lambda x.\ \mathrm{let}\ x_0\ =$$

$$(0,\ \lambda\_.\ x)\ \mathrm{in}\ a^{\bullet}\ [\Gamma, x_0])\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mu\, s\ (\mathrm{inl}\ (\lambda u.\ \mathrm{let}\ u_0\ =\ (0,\ \lambda\_.\ \lambda \kappa.\ \kappa\ \propto\ u)\ \mathrm{in}\ b^{\bullet}\ [\Delta, u_0, y])) \approx \mathrm{let}\ s,\ \mu\ =$$

$$w\ \mathrm{in}\ \mu\, s\ (\mathrm{inl}\ (\lambda u.\ \mathrm{let}\ u_0\ =\ (0,\ \lambda\_.\ \lambda \kappa.\ \kappa\ \propto\ u)\ \mathrm{in}\ \mathrm{let}\ y\ =\ (0,\ \lambda x.\ \mathrm{let}\ x_0\ =\ (0,\ \lambda\_.\ x)\ \mathrm{in}\ a^{\bullet}\ [\Gamma, x_0])\ \mathrm{in}\ b^{\bullet}\ [\Delta, u_0, y])) \overset{def}{=}$$

$$\left(\dfrac{\dfrac{\Delta, u : B, y : A \vdash \mathrm{b} \qquad \Gamma, x : A^{\perp} \vdash \mathrm{a}}{\Gamma, \Delta, u : B \vdash \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\ \textsc{Fuse}}{\Gamma, w : B \,\&\, C, \Delta \vdash \mathrm{let\ inl}\ u = w; \mathrm{fuse}\{y : A \mapsto \mathrm{b}; x : A^{\perp} \mapsto \mathrm{a}\}}\ \&_1\right)^{\bullet}$$

Merge reduction: base cases

$\boxed{\text{merge-n§}}$

$$\left(\dfrac{\dfrac{\Delta, z : A, v : B \vdash \mathrm{a}}{w : \S^{+}{}_{m+n}\, B, \Delta^{m+n}, z : A^{m+n} \vdash \mathrm{traverse}\{w\ \mathrm{as}\ v \mapsto_{m+n} \mathrm{a}\}}\ \S}{w : \S^{+}{}_{m+n}\, B, x : A^m, y : A^n, \Delta^{m+n} \vdash \mathrm{let}\ z = \mathrm{merge}\ x, y; \mathrm{traverse}\{w\ \mathrm{as}\ v \mapsto_{m+n} \mathrm{a}\}}\ \textsc{Merge}_m\right)^{\bullet} \overset{def}{=} \mathrm{let}\ z\ =$$

$$(\mathrm{merge}\ n\ y\ x)\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =\ w\ \mathrm{in}\ \mathrm{schedule}\ (\lambda i.\ \mu\, s\, i\ (\lambda v.\ \mathrm{let}\ v_0\ =\ (0,\ \lambda\_.\ v)\ \mathrm{in}\ \mathrm{let}\ s_0,\ \mu_0\ =\ \Delta\ \mathrm{in}\ (\lambda \delta_0.\ \mathrm{let}\ s_1,\ \mu_1\ =$$

$$z\ \mathrm{in}\ (\lambda z_0.\ a^{\bullet}\ [z_0, v_0, \delta_0])\ (i + s_1,\ \mu_1))\ (i + s_0,\ \mu_0))) \approx \mathrm{let}\ n_0,\ \delta_0\ =\ \Delta\ \mathrm{in}\ \mathrm{let}\ \delta_1\ =\ (m,\ \delta_0)\ \mathrm{in}\ \mathrm{let}\ \delta_2\ =\ (n,\ \delta_0)\ \mathrm{in}\ \mathrm{let}\ s,\ \mu\ =$$

$$w\ \mathrm{in}\ \mu\, s\ (\lambda w_0.\ (\mathrm{schedule}\ (\lambda j.\ \mathrm{let}\ s_0,\ \mu_0\ =\ x\ \mathrm{in}\ (\lambda x_0.\ \mathrm{let}\ s_1,\ \mu_1\ =\ \delta_1\ \mathrm{in}\ (\lambda \delta_3.\ \mathrm{let}\ v\ =$$

$$(0,\ w_0\ j)\ \mathrm{in}\ a^{\bullet}\ [x_0, v, \delta_3])\ (j + s_1,\ \mu_1))\ (j + s_0,\ \mu_0)))\ \gg\ (\mathrm{schedule}\ (\lambda j_0.\ \mathrm{let}\ s_2,\ \mu_2\ =\ y\ \mathrm{in}\ (\lambda y_0.\ \mathrm{let}\ s_3,\ \mu_3\ =\ \delta_2\ \mathrm{in}\ (\lambda \delta_4.\ \mathrm{let}\ v_0\ =$$

$$(0,\ w_0\ (m + j_0))\ \mathrm{in}\ a^{\bullet}\ [y_0, v_0, \delta_4])\ (j_0 + s_3,\ \mu_3))\ (j_0 + s_2,\ \mu_2)))) \overset{def}{=}$$

$$\left(\dfrac{\dfrac{x : A, \Delta, v : B \vdash \mathrm{a} \qquad y : A, \Delta, v : B \vdash \mathrm{a}}{w : \S^{+}{}_{m+n}\, B, x : A^m, y : A^n, \Delta^m, \Delta^n \vdash \mathrm{traverse}\{w\ \mathrm{as}\ v \mapsto_m \mathrm{a}; w\ \mathrm{as}\ v \mapsto_n \mathrm{a}\}}\ \S}{w : \S^{+}{}_{m+n}\, B, x : A^m, y : A^n, \Delta^{m+n} \vdash \mathrm{let}\ \Delta, \Delta = \mathrm{split}_m\ \Delta; \mathrm{traverse}\{w\ \mathrm{as}\ v \mapsto_m \mathrm{a}; w\ \mathrm{as}\ v \mapsto_n \mathrm{a}\}}\ \textsc{Split}_m\right)^{\bullet}$$

$$\boxed{\text{merge-n}\parr}$$

$$\left(\dfrac{\dfrac{\Delta, z : A, v : B \vdash \mathrm{a}}{w : \parr_{m+n} B, \Delta^{m+n}, z : A^{m+n} \vdash \text{coslice } w\{v \mapsto_{m+n} \mathrm{a}\}}\,\parr}{w : \parr_{m+n} B, x : A^m, y : A^n, \Delta^{m+n} \vdash \text{let } z = \text{merge } x, y; \text{coslice } w\{v \mapsto_{m+n} \mathrm{a}\}}\,\textsc{Merge}_m\right)^{\bullet} \stackrel{def}{=} \text{let } z \,=\, (\text{merge}\, n\, y\, x) \text{ in let } s,\ \mu \,=$$

$$w \text{ in } \mu\, s\, (\lambda i.\, \lambda v.\, \text{let } v_0 \,=\, (0,\ \lambda\_.\, \lambda\kappa.\, \kappa \,\propto\, v) \text{ in let } s_0,\ \mu_0 \,=\, \Delta \text{ in } (\lambda\delta_0.\, \text{let } s_1,\ \mu_1 \,=\, z \text{ in } (\lambda z_0.\, a^{\bullet}\, [z_0, v_0, \delta_0])\, (i + s_1,\ \mu_1))\, (i +$$

$$s_0,\ \mu_0)) \approx \text{let } n_0,\ \delta_0 \,=\, \Delta \text{ in let } \delta_1 \,=\, (m,\ \delta_0) \text{ in let } \delta_2 \,=\, (n,\ \delta_0) \text{ in let } s,\ \mu \,=\, w \text{ in } \mu\, s\, (\lambda i.\, \text{if } 0 \,\le\, i \,\wedge\, i \,<\, m \text{ then } \lambda v.\, \text{let } v_0 \,=$$

$$(0,\ \lambda\_.\, \lambda\kappa.\, \kappa \,\propto\, v) \text{ in let } s_0,\ \mu_0 \,=\, x \text{ in } (\lambda x_0.\, \text{let } s_1,\ \mu_1 \,=\, \delta_1 \text{ in } (\lambda\delta_3.\, a^{\bullet}\, [x_0, v_0, \delta_3])\, (i + s_1,\ \mu_1))\, (i + s_0,\ \mu_0) \text{ else } \lambda v_1.\, \text{let } v_2 \,=$$

$$(0,\ \lambda\_.\, \lambda\kappa_0.\, \kappa_0 \,\propto\, v_1) \text{ in let } s_2,\ \mu_2 \,=\, y \text{ in } (\lambda y_0.\, \text{let } s_3,\ \mu_3 \,=\, \delta_2 \text{ in } (\lambda\delta_4.\, a^{\bullet}\, [y_0, v_2, \delta_4])\, (-m + i + s_3,\ \mu_3))\, (-m + i + s_2,\ \mu_2)) \stackrel{def}{=}$$

$$\left(\dfrac{\dfrac{x : A, \Delta, v : B \vdash \mathrm{a} \qquad y : A, \Delta, v : B \vdash \mathrm{a}}{w : \parr_{m+n} B, x : A^m, y : A^n, \Delta^m, \Delta^n \vdash \text{coslice } w\{v \mapsto_m \mathrm{a}; v \mapsto_n \mathrm{a}\}}\,\parr}{w : \parr_{m+n} B, x : A^m, y : A^n, \Delta^{m+n} \vdash \text{let } \Delta, \Delta = \text{split}_m\, \Delta; \text{coslice } w\{v \mapsto_m \mathrm{a}; v \mapsto_n \mathrm{a}\}}\,\textsc{Split}_m\right)^{\bullet}$$

5. Examples

The compiler and the code used for the benchmarks can be obtained from `https://lopezjuan.com/limestone/`.

5.1 Difference operator composed with itself.

Finite differences with wrap-around of an array of type $\bigotimes_{n+1} a$, starting with $4k$ copies of the input.

- Before fusion, allocates intermediate array in memory:

$$
\begin{array}{l}
\text{stencil} \equiv a : *, n : \mathbb{N}, k : \mathbb{N}, p : n \ge 1; i : \bigotimes_{4k} \bigotimes_{n+1} a, o : \parr_k \parr_{n+1} a^{\perp} \vdash \\
\text{cut}\{v : \parr_{2k} \parr_{n+1} a^{\perp} \mapsto \\
\qquad \text{let } i = \text{slice } i; \\
\qquad \text{coslice } v \\
\qquad\quad \{v \mapsto_{2k} \\
\qquad\qquad \text{let } x, y = \text{split}_1\ i;\ \text{let } x = \text{slice } x;\ \text{let } x, y = \text{split}_1\ x; \\
\qquad\qquad \text{let } y = \text{slice } y;\ \text{let } y, y = \text{split}_1\ y; \\
\qquad\qquad \text{coslice } v\{v \mapsto_1 (-)[x, y, v];\ v \mapsto_n (-)[y, y, v]\}\} \\
\quad\ w : \bigotimes_{2k} \bigotimes_{n+1} a \mapsto \\
\qquad \text{let } w = \text{slice } w; \\
\qquad \text{coslice } o \\
\qquad\quad \{o \mapsto_k \\
\qquad\qquad \text{let } x, y = \text{split}_1\ w;\ \text{let } x = \text{slice } x;\ \text{let } x, y = \text{split}_1\ x; \\
\qquad\qquad \text{let } y = \text{slice } y;\ \text{let } y, y = \text{split}_1\ y; \\
\qquad\qquad \text{coslice } o\{o \mapsto_1 (-)[x, y, o];\ o \mapsto_n (-)[y, y, o]\}\}\}
\end{array}
$$

- After fusion, it does not allocate an intermediate array in memory:

$$
\begin{array}{l}
\text{stencil} \equiv a : *, n : \mathbb{N}, k : \mathbb{N}, p : n \ge 1; i : \bigotimes_{4k} \bigotimes_{n+1} a, o : \parr_k \parr_{n+1} a^{\perp} \vdash \\
\text{let } i = \text{slice } i;\ \text{let } x, y = \text{split}_{2k}\ i; \\
\text{coslice } o \\
\quad \{o \mapsto_k \text{let } x, x = \text{split}_1\ x;\ \text{let } y, y = \text{split}_1\ y; \\
\qquad\quad \text{let } x = \text{slice } x;\ \text{let } x, y = \text{split}_1\ x;\ \text{let } x = \text{slice } x; \\
\qquad\quad \text{let } x, y = \text{split}_1\ x;\ \text{let } y = \text{slice } y;\ \text{let } y, y = \text{split}_1\ y; \\
\qquad\quad \text{let } y = \text{slice } y;\ \text{let } y, y = \text{split}_1\ y; \\
\qquad\quad \text{coslice } o\{o \mapsto_1 \text{cut}\{v : a^{\perp} \mapsto (-)[x, y, v] \\
\qquad\qquad\qquad\qquad\qquad\quad v : a \mapsto \text{cut}\{v : a \mapsto (-)[v, v, o] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad v : a^{\perp} \mapsto (-)[x, y, v]\}\} \\
\qquad\qquad\qquad o \mapsto_n \text{cut}\{v : a^{\perp} \mapsto (-)[y, y, v] \\
\qquad\qquad\qquad\qquad\qquad\quad v : a \mapsto \text{cut}\{v : a \mapsto (-)[v, v, o] \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad v : a^{\perp} \mapsto (-)[y, y, v]\}\}\}\}
\end{array}
$$

5.2 FFT

FFT implementation before fusion:

```
fftStep ≡ n : ℕ, C : ∗; i : ⨂_2n C, o : ⅋_2n C⊥ ⊢
cut{v : ⅋_n C⊥ ⅋ ⅋_n C⊥ ↦
        let i = slice i; let x, y = split_n i;
        connect v to{τ ↦ coslice τ{τ ↦_n x ↔ τ}
                     σ ↦ coslice σ{σ ↦_n y ↔ σ}}
    w : ⨂_n C ⊗ ⨂_n C ↦
        cut{v : ⅋_n(C ⊸ C⊥) ↦
                let τ, τ = w; let τ = slice τ; let τ = slice τ;
                coslice v
                  {v ↦_n connect v to{τ ↦ τ ↔ τ; σ ↦ τ ↔ σ}}
            w : ⨂_n(C ⊗ C) ↦
                cut{v : ⅋_n(C⊥ ⊗ C⊥) ↦
                        let w = slice w;
                        coslice v{v ↦_n let τ, τ = w; let τ, τ = v;
                                        bff[τ, τ, τ, τ]}
                    w : ⨂_n(C ⅋ C) ↦
                        cut{v : ⨂_n(C⊥ ⊗ C⊥) ↦
                                let w = slice w; let v = slice v;
                                traverse{↦_n w ↔ v}
                            w : ⅋_n(C ⅋ C) ↦
                                cut{v : ⨂_n C⊥ ⊗ ⨂_n C⊥ ↦
                                        let τ, τ = v; let τ = slice τ;
                                        let τ = slice τ;
                                        coslice w
                                          {w ↦_n connect w to
                                                 {τ ↦ τ ↔ τ
                                                  σ ↦ τ ↔ σ}}
                                    w : ⅋_n C ⅋ ⅋_n C ↦
                                        cut{v : ⨂_2n C⊥ ↦
                                                let v = slice v;
                                                let x, y = split_n v;
                                                connect w to
                                                  {τ ↦
                                                      coslice τ
                                                        {τ ↦_n x ↔ τ}
                                                   σ ↦
                                                      coslice σ
                                                        {σ ↦_n
                                                           y ↔ σ}}
                                            w : ⅋_2n C ↦
                                                sync{vf : C⊥^2n ↦
                                                          coslice w
                                                            {w ↦_2n
                                                               vf ↔ w}
                                                     wf : C^2n ↦
                                                          coslice o
                                                            {o ↦_2n
                                                               wf ↔ o}}}}}}}}
```

FFT implementation after fusion:

```
fftStep ≡ n : ℕ, C : ∗; i : ⨂_2n C, o : ⅋_2n C⊥ ⊢
let i = slice i; let x, y = split_n i;
cut{w : C ⅋ C^n ↦
        sync{vf : C⊥^2n ↦
                  let x, y = split_n vf;
                  traverse
                    {↦_n connect w to{τ ↦ x ↔ τ; σ ↦ y ↔ σ}}
              wf : C^2n ↦ coslice o{o ↦_2n wf ↔ o}}
    v : C⊥ ⊗ C⊥ ↦ let τ, τ = v; bff[x, y, τ, τ]}
```

5.3 Wave stencil

We decompose the algorithm using a linear version of standard arrow combinators [1]. With them, streaming computations can be expressed as morphisms and products in a suitable category. The resulting data flow is shown in Fig. 1.

(***) : $\S_n(A \multimap B),\ \S_n(C \multimap D),\ (\S_n(A \otimes C) \multimap B \otimes D)^{\perp} \vdash$
(&&&) : $\S_n(D \multimap B),\ \S_n(D \multimap C),\ (\S_n(D \multimap B \otimes C))^{\perp} \vdash$
(>>>) : $\S_n(A \multimap B),\ \S_n(B \multimap C),\ \S_n(A \multimap C) \vdash$
(++) : $\S_n A,\ \S_m A,\ \S_{m+n} A \vdash$
arrowSeq n : $(A \multimap B)^n,\ (\S_n(A \multimap B))^{\perp} \vdash$
delaySeq k n : $\bigotimes_{k-1} D,\ (\S_n(D \multimap D))^{\perp} \vdash$
windowSeq k n : $\bigotimes_{k-1} D,\ (\S_n(D \multimap \bigotimes_k D))^{\perp} \vdash$
coapplySeq : $\S_n(A \multimap B),\ \S_n B^{\perp},\ \S_n A \vdash$
applySeq : $\S_n(A \multimap B),\ \S_n A,\ \S_n A^{\perp} \vdash$

waveSymm ≡prev : $\S_n(\mathbb{R} \otimes \mathbb{R})$, next : $\S_n(\mathbb{R}^{\perp} \parr \mathbb{R}^{\perp}) \vdash$
  cut { arrowSeq 1 "waveL" } wL ↦
  cut { arrowSeq (n − 2) "waveC" } wC ↦
  cut { arrowSeq 1 "waveR" } wR ↦
  cut { arrowSeq (n + 1) { (_,p1) ↦ p1 } } snd ↦
  cut { arrowSeq n { a b ↦ a ↔ b } } identity ↦

  cut { wL ++ wC ++ wR } waveF : $\S_n(\bigotimes_3 \mathbb{R} \otimes \mathbb{R}) \multimap \mathbb{R}$ ↦
  cut { identity *** waveF } waveStep ↦
  cut { coapplySeq waveStep next } next' ↦
  cut { delaySeq 1 (n+1) ⊗[0] } delay ↦
  cut { windowSeq 3 (n+1) ⊗[0,0] } window ↦
  cut { delay *** window } stencil ↦
  cut { snd &&& stencil } pipeline ↦
  applySeq pipeline (prev ++ [0]) ({x ↦ dump x} ++ next)

Figure 1: Wave stencil as a data flow. Cuts where the variable of the left side is immediately used as the last argument of a call to another derivation are omitted for conciseness, and literals are used in the place of certain types.

5.4 QuickHull

We include the implementation two combinators from the implementation of the QuickHull example (Fig. 2 and Fig. 3). The full details are available in the code release.

References

[1] J. Hughes. Generalising monads to arrows. *Science of computer programming*, 37(1):67–111, 2000.

$\text{split} \equiv \mathit{pred} : ((B \multimap 1 \oplus 1) \,\&\, 1)^n, \mathit{input} : \S_n(A \otimes B \oplus 1), \mathit{output} : \S_n(A^\perp \,\&\, \bot), \mathit{output}_1 : \S_n(A^\perp \,\&\, \bot) \vdash$

$\quad \text{traverse}$

$\quad\quad \{\mathit{input} \text{ as } \mathit{input}_1, \mathit{output} \text{ as } \mathit{output}_2, \mathit{output}_1 \text{ as } \mathit{output}_3 \mapsto_n$

$\quad\quad\quad \text{case } \mathit{input}_1 \text{ of}$

$\quad\quad\quad\quad \{\text{inl } \mathit{input}_2 \mapsto$

$\quad\quad\quad\quad\quad \text{let } \tau, \tau_1 = \mathit{input}_2;\ \text{let inl } \mathit{pred}_1 = \mathit{pred};$

$\quad\quad\quad\quad\quad \text{connect } \mathit{pred}_1 \text{ to}$

$\quad\quad\quad\quad\quad\quad \{\tau_2 \mapsto \tau_1 \leftrightarrow \tau_2$

$\quad\quad\quad\quad\quad\quad\ \sigma \mapsto \text{case } \sigma \text{ of}\{\text{inl } \sigma_1 \mapsto$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{let } \diamond = \sigma_1;$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{mix}\{\text{let inl } \mathit{output}_4 = \mathit{output}_2;$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \tau \leftrightarrow \mathit{output}_4$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{let inr } \mathit{output}_5 = \mathit{output}_3;$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{yield to } \mathit{output}_5\}$

$\quad\quad\quad\quad\quad\quad\quad \text{inr } \sigma_2 \mapsto$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{let } \diamond = \sigma_2;$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{mix}\{\text{let inl } \mathit{output}_6 = \mathit{output}_3;$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \tau \leftrightarrow \mathit{output}_6$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{let inr } \mathit{output}_7 = \mathit{output}_2;$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{yield to } \mathit{output}_7\}\}\}$

$\quad\quad\quad\quad \text{inr } \mathit{input}_3 \mapsto$

$\quad\quad\quad\quad\quad \text{let } \diamond = \mathit{input}_3;\ \text{let inr } \mathit{pred}_2 = \mathit{pred};$

$\quad\quad\quad\quad\quad \text{let } \diamond = \mathit{pred}_2;$

$\quad\quad\quad\quad\quad \text{mix}\{\text{let inr } \mathit{output}_8 = \mathit{output}_2;\ \text{yield to } \mathit{output}_8$

$\quad\quad\quad\quad\quad\quad \text{let inr } \mathit{output}_9 = \mathit{output}_3;\ \text{yield to } \mathit{output}_9\}\}\}$

(a) CLL term

```
splitDeriv  = D0 $ runFreshM $ do
  n <- freshFrom "n";  let  szN = var  n
  a <- freshFrom "A";  let  tyA = var  a
  b <- freshFrom "B";  let  tyB = var  b

  pred <- freshFrom "pred"
  input  <- freshFrom "input"
  [1output , 2output]  <- mapM freshFrom ["output","output"]

  Deriv  " split "  (SeqCtx
     [n    :::.    T TSize
     ,a    :::.    T TType
     ,b    :::.    T TType
     ]
     [pred     :::    (tyB ⊸  bool)  &: one  :^  szN
     , input    :::.   bigSeq szN (option  (tyA ⊗  tyB))
     , 1output   :::.   dual  (bigSeq szN (option  tyA))
     , 2output   :::.   dual  (bigSeq szN (option  tyA))
     ])  <$>
    CLL.bigseq [  seqstep  szN [input , 1output , 2output]  $  \[ input , 1output , 2output]  ->
       plusOption  input  $ \case
          Nothing -> ignoreLazy pred $ CLL.mix' [withR 1output  $  yield
                                                ,withR 2output  $  yield
                                                ]
          Just  input  ->
             tensor  input  $ \a  k  ->
                forceLazy  pred $ \pred ->
                     apply  pred [k]  $ \b  ->
                        whether  b $ \case
                           True   -> CLL.mix' [withL 1output  (axiom a)
                                              ,withR 2output  $  yield
                                              ]
                           False   -> CLL.mix' [withL 2output  (axiom a)
                                               ,withR 1output  $  yield
                                               ]
       ]
```

(b) Haskell DSL program

Figure 2: The split combinator splits a sequence of pairs into two sequences, based on a predicate applied to the second element of the pair. By requiring control over the two output sequences, the need for intermediate storage is avoided.

$$
\begin{array}{l}
\mathrm{dotSeq} \equiv a : \bigotimes_n \mathbb{R}, b : \bigotimes_n \mathbb{R}, r : \mathbb{R}^{\perp} \vdash \\
\mathrm{fuse}\{\tau : \text{⅋}_n \mathbb{R}^{\perp} \mapsto \tau \leftrightarrow \mathrm{zipWith}(*)[a, b] \\
\quad \tau_1 : \bigotimes_n \mathbb{R} \mapsto \\
\qquad \mathrm{let}\ \tau_2 = \mathrm{slice}\ \tau_1; \\
\qquad \mathrm{loop}\{mw : \mathbb{R}^{\perp} \mapsto mw \leftrightarrow 0.0 \\
\qquad\qquad mu : \S_{n+1}(\mathbb{R} \otimes (\mathbb{R}^{\perp} \,\&\, 1)) \mapsto \\
\qquad\qquad\quad \mathrm{traverse}\{mu \text{ as } mu_1 \mapsto_n \\
\qquad\qquad\qquad\qquad \mathrm{let}\ \tau_3, \tau_4 = mu_1;\ \mathrm{let\ inl}\ \tau_5 = \tau_4; \\
\qquad\qquad\qquad\qquad \tau_5 \leftrightarrow +[\tau_3, \tau_2] \\
\qquad\qquad\qquad\quad mu \text{ as } mu_2 \mapsto_1 \\
\qquad\qquad\qquad\qquad \mathrm{let}\ \tau_6, \tau_7 = mu_2;\ \mathrm{let\ inr}\ \tau_8 = \tau_7; \\
\qquad\qquad\qquad\qquad \mathrm{let}\ \diamond = \tau_8;\ r \leftrightarrow \tau_6\}\}\}
\end{array}
$$

(a) CLL term

```
dotSeqDeriv :: Deriv0 Id Id
dotSeqDeriv = D0 $ runFreshM $ do
  [n,a,b,r] <- mapM freshFrom ["n","a","b","r"]
  let szN = var n
  Deriv "dotSeq"
        SeqCtx {
             _intuit  = [n  :::.  T TSize]
           , _linear  = [a  :::.  bigTensor szN tDouble
                        ,b  :::.  bigTensor szN tDouble
                        ,r  :::.  dual (T PDouble)]
           }
      <$>
           ( ffiCall  (bigTensor szN tDouble) "zip∗()" [a,b] $ \c -> slice c $ \c ->
                seqsync0 (tDouble :^ 1) (1 + szN)
                 (\z -> ffiLit ' ("0.0" ::: tDouble) z)
                 (\s ->
                      CLL.bigseq [
                          seqstep szN [s] $ \[s] -> foldYield s tDouble $
                                               \prev next -> (prev `fPlusD`` c $ next)

                         , seqstep 1    [s] $ \[s] -> foldIdentity  s tDouble (axiom r)
                          ]))
```

(b) Haskell DSL version

Figure 3: Combinator dot. It demonstrates the usage of LOOP to accumulate intermediate results; in this case, a sum.